

# Quasi-periodic Solutions of a Derivative Nonlinear Schrödinger Equation

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## Abstract

This paper is concerned with a one dimensional (1D) derivative nonlinear Schrödinger equation with periodic boundary conditions

$$iu_t + u_{xx} + i|u|^2u_x = 0, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}.$$

We show that above equation admits a family of real analytic quasi-periodic solutions with two Diophantine frequencies. The proof is based on a partial Birkhoff normal form and KAM method.

**Keywords.** Derivative nonlinear Schrödinger equation, Quasi-periodic solution, KAM theory.

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## 1 Introduction and Main Result

In this paper, we consider the derivative nonlinear Schrödinger equation (Chen-Lee-Liu-equation [6])

$$iu_t + u_{xx} + i|u|^2u_x = 0 \tag{1.1}$$

with periodic boundary condition

$$u(t, 0) = u(t, 2\pi), \tag{1.2}$$

which appears in studies of ultrashort optical pulses. Moreover, Eq. (1.1) has several applications in e.g. plasma physics and nonlinear fiber optics referring to [15] and [26].

Consider the Hamiltonian partial differential equation

$$\dot{w} = Aw + F(w).$$

For some Sobolev space  $\mathcal{H}^p \ni w$ , linear operator  $A$  maps  $\mathcal{H}^p$  to  $\mathcal{H}^{p-d}$  and nonlinear term  $F$  sends some neighborhood of  $\mathcal{H}^p$  to  $\mathcal{H}^{p-\delta}$ . One calls  $d$  and  $\delta$  the orders of  $A$  and  $F$  respectively.

When  $\delta \leq 0$ , the vector field  $F$  is called bounded perturbation. The existence of quasi-periodic solutions of such PDEs has been widely investigated by many authors [2, 4, 5, 7–11, 13, 16–19, 27, 28, 30, 32, 34, 35, 37].

When  $\delta > 0$ , the vector field  $F$  is called unbounded perturbation. Unlike the bounded case, there are few results of KAM theory for partial differential equations with unbounded perturbation. The first KAM theorem for unbounded perturbations is due to Kuksin

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[20, 21] under the assumption  $0 < \delta < d - 1$ . See also Kappeler and Pöschel [22]. Another KAM theorem with unbounded linear Hamiltonian perturbation is due to Bambusi and Graffi [1] which consider the time dependent linear Schrödinger equation.

When  $0 < \delta = d - 1$ , which is called “the limiting case”, the nonlinearity of the PDE is the strongest. Recently, Liu and Yuan [23] give a theorem which generalizes Kuksin’s theorem from  $\delta < d - 1$  to  $\delta \leq d - 1$ . In their paper, they still consider the homological equations of variable coefficients:

$$-i\partial_\omega u + \lambda u + \mu(\theta)u = p(\theta), \quad |\text{Im } \theta| < s, \quad (1.3)$$

Using the generalized Kuksin’s theorem, Liu and Yuan [24] establish an improved KAM theorem which can prove the existence of quasi-periodic solution of a class of derivative nonlinear Schrödinger equations (DNLS)

$$iu_t + u_{xx} - M_\xi u + i f(u, \bar{u})u_x = 0, \quad (1.4)$$

with Dirichlet boundary conditions, where  $f(u, \bar{u})$  be a analytic function in  $\mathbb{C}^2$  with

$$\overline{f(u, \bar{u})} = f(u, \bar{u}), f(-u, -\bar{u}) = -f(u, \bar{u}).$$

Then, Geng and Wu [12] consider the derivative nonlinear Schrödinger equation

$$iu_t - u_{xx} - i(|u|^4 u)_x = 0 \quad (1.5)$$

with periodic boundary condition. Unlike [24], by using the compact form and the gauge invariant property, the homological equation (1.3) becomes into the following forms:

$$-i\partial_\omega u + \lambda u = p(\theta). \quad (1.6)$$

Since normal form obtained in [12] is independent of the angle variables  $\theta$ , it is different from Kuksin’s theorem [21] and Liu and Yuan’s theorem [24]. Then, using an abstract KAM theorem with angle independent normal form, they obtain the real analytic quasi-periodic solutions for the derivative nonlinear Schrödinger equation (1.5) with only two Diophantine frequencies.

Lately, for a class of derivative nonlinear Schrödinger equation

$$iu_t + u_{xx} + i(f(|u|^2)u)_x = 0, \quad (1.7)$$

Liu and Yuan [25] prove that Eq.(1.7) with periodic boundary conditions admits many  $C^\infty$  (not real analytic) quasi-periodic solutions with  $N$  Diophantine frequencies, where  $N$  is any positive integer. It is worth to note that the momentum conservation plays an important role in their results. To use both Kuksin’s lemma in [20] and the estimates in [23], the homological equations must be scalar i.e. the normal frequency  $\Omega_j$  is required to be simple  $\Omega_j^\sharp = 1$ . So the KAM theorem for unbounded perturbations in [24] can not be used to the derivative nonlinear Schrödinger equation(1.7) with periodic boundary conditions, since the multiplicity  $\Omega_j^\sharp = 2$ . But this difficulty can be avoided since the nonlinear  $i(f(|u|^2)u)_x$  does not contain the space variable  $x$  explicitly, so that momentum is conserved for (1.7). More details can be found in [25].

In this paper, we consider the derivative nonlinear Schrödinger equation (1.1)

$$iu_t + u_{xx} + i|u|^2 u_x = 0$$

with periodic boundary condition. Obviously, Eq. (1.1) is not contained in Eq. (1.4), which is our first motivation to consider the quasi-periodic solutions of (1.1).

For (1.7), if  $f$  is the identity function, i.e.  $f(z) = z$ , then (1.7) reduces to

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0, \quad (1.8)$$

which appears in various physical applications and has been widely studied in the literature. Applying the gauge transformation ((2.12) in [31])

$$v = u(x) \exp \left\{ -\frac{i}{2} \int_{-\infty}^x |u(\eta)|^2 d\eta \right\},$$

above Eq. (1.8) is transformed into Eq. (1.1). However, in [33], they point that

“But the gauge transformation can’t preserve the reduction conditions in spectral problem of the Kaup and Newell (KN) [14] system and involve complicated integrations. So it deserves to be investigated separately.”

This is our second motivation to consider (1.1).

To obtain the real analytic quasi-periodic solutions of Eq. (1.1), we construct a KAM iteration for Hamiltonian PDEs with some special perturbations which admits the compact form and the gauge invariant property like [12].

Assume that

$$[u] := \frac{1}{2\pi} \int_0^{2\pi} u dx = 0, \quad (1.9)$$

then the main result is described as follows:

**Theorem 1.1.** *Consider the derivative nonlinear Schrödinger equation (1.1) with periodic boundary conditions (1.2) and (1.9). Fix  $n_1, n_2$  satisfying that  $n_1$  is odd and  $|n_2 - n_1| = 4$ . Then there exists a Cantor subset  $\mathcal{O}_* = \mathcal{O}_*(n_1, n_2) \subset \mathbb{R}_+^2$  of positive Lebesgue measure, such that each  $\xi \in \mathcal{O}_*$  corresponds to a real analytic, quasi-periodic solution*

$$u(t, x) = \sum_{j=1}^2 \sqrt{\frac{1}{2\pi}} \xi_j e^{i(\omega_{*j} t + n_j x)} + O(|\xi|^{\frac{3}{2}})$$

of (1.1), (1.2), (1.9) with two Diophantine frequencies

$$\omega_{*j} = n_j^2 + O(|\xi|), \quad 1 \leq j \leq 2.$$

Moreover, the quasi-periodic solutions  $u$  are linearly stable and depend on  $\xi$  Whitney smoothly.

**Remark 1.1.** *Note that the solution which we obtain in Theorem 1.1 is real analytic like [12], although the number of frequencies is only 2 not any positive integer  $N$ . The reason is that in the KAM iteration, we still let  $s_m$ , the radius of  $\text{Im } \theta$ , such that  $s_m \rightarrow \frac{s}{2}$  as  $m \rightarrow \infty$ .*

The rest of the paper is organized as follows: In section 2, we give some definition such as compact form and gauge invariant property. Although all the definitions are the same as [12], we would like to list them here for reader’s convenience. In Section 3, we will give the Hamiltonian setting corresponding to the Eq. (1.1) and derive a partial Birkhoff normal form of order four for the lattice Hamiltonian. In Section 4, we will show some conditions about frequencies and perturbation for the lattice Hamiltonian obtained in Section 3. In Section 5, we will give details for one step KAM iteration, summarize as an iteration lemma and prove its convergence. At last, we give the necessary measure estimate for the parameter set. Some technical lemmas necessary are given in the Appendix.

## 2 Preliminary

Denote  $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$ , for any integer  $a > 0$  and  $p \geq 0$ , we introduce the phase space, complex valued functions space on  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ :

$$\mathcal{H}^{a,p} = \left\{ u \in L^2(\mathbb{T}, \mathbb{C}) : \|u\|_{a,p}^2 = \sum_{n \in \mathbb{Z}_*} |\hat{u}_n|^2 |n|^{2p} e^{2a|n|} < +\infty \right\},$$

where  $u = \sum_{n \in \mathbb{Z}_*} \hat{u}_n e^{inx}$  is the discrete Fourier transform.

Let  $\ell^{a,p}$  be the space of all bi-complex valued sequences  $q = (\dots, q_{-2}, q_{-1}, q_1, q_2, \dots)$  with

$$\|q\|_{a,p}^2 = \sum_{n \in \mathbb{Z}_*} |q_n|^2 |n|^{2p} e^{2a|n|} < +\infty.$$

The convolution  $w * z$  of two such sequences is defined by  $(w * z)_n = \sum_m w_{n-m} z_m$ .

**Lemma 2.1.** [17] *For  $a > 0$ ,  $p > \frac{1}{2}$ , the space  $\ell^{a,p}$  is a Banach algebra with respect to convolution of sequences, and*

$$\|w * z\|_{a,p} \leq c \|w\|_{a,p} \|z\|_{a,p},$$

with a constant  $c$  depending only on  $p$ .

In the following, we give the same definitions of compact form and gauge invariant property in [12]. To keep the continuity and enhance the readability, we list corresponding definitions and properties here.

Let

$$\mathcal{J} = \{\{n_1, n_2\} \in \mathbb{Z}_* \mid n_1 \text{ is odd and } |n_2 - n_1| = 4\}.$$

Without loss of generality, we assume that  $n_2 > n_1 > 0$  for simplicity.

**Definition 2.1.** [12] *Given  $\{n_1, n_2\} \in \mathcal{J}$ . A real analytic function*

$$F = F(\theta, I, z, \bar{z}) = \sum_{k, \alpha, \beta} F_{k, \alpha, \beta} e^{i(k, \theta)} z^\alpha \bar{z}^\beta$$

*is said to admit a **compact form** with respect to  $n_1, n_2$  if*

$$F_{k, \alpha, \beta} = 0, \quad \text{whenever } k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n) n \neq 0,$$

*where  $k = (k_1, k_2) \in \mathbb{Z}^2$  and  $\alpha = (\dots, \alpha_n, \dots), \beta = (\dots, \beta_n, \dots), \alpha_n, \beta_n \in \mathbb{N}$ , with finitely many nonzero components of positive integers.*

Consider the Poisson bracket

$$\{F, G\} = \sum_{1 \leq j \leq 2} \frac{\partial F}{\partial \theta_j} \frac{\partial G}{\partial I_j} - \frac{\partial F}{\partial I_j} \frac{\partial G}{\partial \theta_j} + i \sum_{j \in \mathbb{Z}} \frac{\partial F}{\partial z_j} \frac{\partial G}{\partial \bar{z}_j} - \frac{\partial F}{\partial \bar{z}_j} \frac{\partial G}{\partial z_j},$$

we have the following lemma

**Lemma 2.2.** [12] *Given  $\{n_1, n_2\} \in \mathcal{J}$  and consider two real analytic functions  $F(\theta, I, z, \bar{z}), G(\theta, I, z, \bar{z})$ . If both  $F$  and  $G$  have compact forms with respect to  $n_1, n_2$ , then so does  $\{F, G\}$ .*

**Definition 2.2.** [12] A real analytic function

$$F = F(\theta, I, z, \bar{z}) = \sum_{k, \alpha, \beta} F_{k, \alpha, \beta} e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta$$

is said to admit ***gauge invariant property*** if

$$F_{k, \alpha, \beta} = 0, \quad \text{whenever } k_1 + k_2 + \sum_n (\alpha_n - \beta_n) \neq 0,$$

where  $k = (k_1, k_2) \in \mathbb{Z}^2$  and  $\alpha = (\cdots, \alpha_n, \cdots), \beta = (\cdots, \beta_n, \cdots), \alpha_n, \beta_n \in \mathbb{N}$ , with finitely many nonzero components of positive integers.

**Lemma 2.3.** [12] Consider two real analytic functions  $F(\theta, I, z, \bar{z}), G(\theta, I, z, \bar{z})$ . If both  $F$  and  $G$  admit the gauge invariant property, then so does  $\{F, G\}$ .

**Lemma 2.4.** [12] Given  $\{n_1, n_2\} \in \mathcal{J}$  and consider a real analytic functions  $F(\theta, I, z, \bar{z})$ . If  $F$  has compact form and admits gauge invariant property with respect to  $n_1, n_2$ , then  $F$  contains no terms of the form  $e^{i\langle k, \theta \rangle} z_n \bar{z}_n$  with  $k \neq 0$  and  $e^{i\langle k, \theta \rangle} z_n \bar{z}_m$  with  $k = 0$  and  $n \neq m$ .

Although this lemma has been proved in [12], in order to make the reader understand the role which compact form and gauge invariant property play in KAM iteration, we would like to “prove” it again.

*Proof.* Consider  $F(\theta, I, z, \bar{z}) = \sum_{k, \alpha, \beta} F_{k, \alpha, \beta} e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta$  with  $\alpha = \beta = e_n$ , where  $e_n$  denotes the  $n$ -th component being 1 and the other components being 0. Since  $F$  has compact form with respect to  $n_1, n_2$  and admits the gauge invariant property, we have

$$\begin{cases} k_1 n_1 + k_2 n_2 + n - n &= 0, \\ k_1 + k_2 + 1 - 1 &= 0. \end{cases}$$

In View of  $n_1 \neq n_2$ , we obtain that  $k_1 = k_2 = 0$ . Then consider  $F(\theta, I, z, \bar{z})$  with  $\alpha = e_n, \beta = e_m$  and  $k = 0$ . Since  $F$  has compact form with respect to  $n_1, n_2$ , we obtain

$$n - m = 0.$$

Hence, lemma is proved.  $\square$

**Remark 2.1.** From the proof of Lemma 2.4, we can find that the method in [12] restrict the number of frequencies of quasi-periodic solution to only 2, unlike in [25] any positive number  $N$ .

**Remark 2.2.** We should show that compact form and the gauge invariant property will be preserved along KAM iterations. These properties enable simplify the homological equation in each KAM step.

We denote

$$\mathcal{A}_{n_1, n_2} = \left\{ P : P = \sum_{k \in \mathbb{Z}^2, l \in \mathbb{N}^2, \alpha, \beta} P_{k, l, \alpha, \beta} e^{i\langle k, \theta \rangle} I^l z^\alpha \bar{z}^\beta \right\},$$

where  $k, \alpha, \beta$  have the following relations:

$$k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n) n = 0 \text{ and } k_1 + k_2 + \sum_n (\alpha_n - \beta_n) = 0.$$

### 3 Hamiltonian and Normal Form

In this section, We will study (1.1) as a Hamiltonian system on some suitable phase space  $\mathcal{P}$ . Using the Hamiltonian formulation, we rewrite (1.1) with periodic boundary condition in the Hamiltonian form

$$u_t = -i \frac{\partial H}{\partial \bar{u}}, \quad (3.1)$$

with Hamiltonian

$$H = \int_0^{2\pi} |u_x|^2 dx - \frac{i}{2} \int_0^{2\pi} |u|^2 \bar{u} u_x dx, \quad (3.2)$$

where the gradient is defined with respect to inner product in  $L^2$ :  $\langle u, v \rangle = \int_0^{2\pi} u \bar{v} dx$ .

Consider operator  $A = -\partial_{xx}$  with the periodic boundary condition. The eigenfunctions is  $\{\phi_j(x) = \sqrt{\frac{1}{2\pi}} e^{ijx}\}$  and corresponding eigenvalue is  $\lambda_j = j^2$ .

To write it in infinitely many coordinates, we make the ansatz

$$u = \mathcal{L}q = \sum_{j \in \mathbb{Z}_*} q_j(t) \phi_j(x). \quad (3.3)$$

The coordinates are taken from Hilbert space  $\ell^{a,p}$ . Due to the definition of spaces, there is an isomorphism  $\mathcal{L} : \ell^{a,p} \mapsto \mathcal{H}^{a,p}$  with  $\|u\|_{a,p}^2 = \|\bar{u}\|_{a,p}^2 = \|q\|_{a,p}^2$ , for each  $p \geq 0$ .

Fixed  $a > 0$  and  $p > \frac{3}{2}$  in the following, one obtains the Hamiltonian

$$H = \Lambda + G \quad (3.4)$$

with

$$\begin{aligned} \Lambda &= \sum_{j \in \mathbb{Z}_*} \lambda_j |q_j|^2, \\ G &= -\frac{i}{2} \int_0^{2\pi} |\mathcal{L}q|^2 (\overline{\mathcal{L}q})(\mathcal{L}q)_x dx, \end{aligned}$$

on the phase space  $\ell^{a,p}$  with symplectic structure  $-i \sum_{j \in \mathbb{Z}_*} dq_j \wedge d\bar{q}_j$ . Its equation of motion are

$$\dot{q}_j = -i \frac{\partial H}{\partial \bar{q}_j}, \quad j \in \mathbb{Z}_*. \quad (3.5)$$

They are the classical Hamiltonian equation of motion for the real and imaginary parts of  $q_j = x_j + iy_j$  written in complex notion.

**Lemma 3.1.** *Let  $a > 0$  and  $p \geq 0$ . If a curve  $\mathbb{R} \rightarrow \ell^{a,p}$ ,  $t \rightarrow q(t)$  is a real analytic solution of (3.5), then*

$$u = \mathcal{L}q = \sum_{j \in \mathbb{Z}_*} q_j(t) \phi_j(x).$$

*is a solution of (1.1) that is real analytic on  $\mathbb{R} \times [0, 2\pi]$ .*

*Proof.* The proof is similar to Lemma 1 in [19], we omit it.  $\square$

Then we establish the regularity of nonlinear Hamiltonian vector field  $X_G$ . The perturbation term  $G$  has the following properties:

**Lemma 3.2.** *For  $a > 0$  and  $p > \frac{3}{2}$ , the function  $G$  is analytic in some neighborhood of the origin in  $\ell^{a,p}$  with real value, and  $G_{\bar{q}}$  is an analytic map from some neighborhood of the origin in  $\ell^{a,p}$  into  $\ell^{a,p-1}$  with*

$$\|G_{\bar{q}}\|_{a,p-1} = O(\|q\|_{a,p}^3). \quad (3.6)$$

*Proof.* Let  $G_{\bar{q}} = (\{\frac{\partial G}{\partial \bar{q}_l}\})$ , where

$$\frac{\partial G}{\partial \bar{q}_l} = -i \int_0^{2\pi} |u|^2 u_x \bar{\phi}_l dx, \quad u = \mathcal{L}q.$$

Let  $q$  be in  $\ell^{a,p}$ , then  $(jq_j)_{j \in \mathbb{Z}_*} \in \ell^{a,p-1}$ . By the algebra property, we can get

$$\| |u|^2 u_x \|_{a,p-1} \leq c \|u\|_{a,p}^3.$$

The components of the gradient  $G_{\bar{q}}$  are its Fourier coefficients, so  $G_{\bar{q}}$  in  $\ell^{a,p-1}$ , with

$$\|G_{\bar{q}}\|_{a,p-1} \leq \| |u|^2 u_x \|_{a,p-1} \leq c \|u\|_{a,p}^3 \leq c \|q\|_{a,p}^3.$$

The regularity of  $G_{\bar{q}}$  follows from the regularity of its components. □

For the nonlinearity  $i|u|^2 u_x$ , we find

$$G = \frac{1}{2} \sum_{i,j,k,l} j G_{ijkl} q_i q_j \bar{q}_k \bar{q}_l = \sum_{\alpha,\beta} G_{\alpha,\beta} q^\alpha \bar{q}^\beta,$$

where

$$G_{ijkl} = \int_0^{2\pi} \phi_i \phi_j \bar{\phi}_k \bar{\phi}_l dx = \begin{cases} \frac{1}{2\pi}, & \text{if } i+j = k+l, \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 3.1.** From above special forms of  $G$  and  $G_{ijkl}$ , we know that  $G \in \mathcal{A}_{n_1, n_2}$ , i.e.  $G_{\alpha,\beta} \neq 0$  when  $\sum_n (\alpha_n - \beta_n)n = 0$  and  $\sum_n (\alpha_n - \beta_n) = 0$ .

**Lemma 3.3.** If  $i+j = k+l$  and  $\{i, j\} \neq \{k, l\}$ , then

$$\lambda_i + \lambda_j - \lambda_k - \lambda_l = i^2 + j^2 - k^2 - l^2 \neq 0.$$

*Proof.* Suppose  $i^2 + j^2 = k^2 + l^2$ , we can get  $ij = kl$ . Since there are two real roots for quadratic polynomial at most, we can get  $\{i, j\} = \{k, l\}$ . This is a contradiction. □

For all indices  $i, j, k, l$  satisfying  $i+j = k+l$ , we denote

$$\mathcal{N} = \{(i, j, k, l) \in \mathbb{Z}_*^4 \mid \{i, j\} = \{k, l\}\},$$

$$\Delta_l = \{(i, j, k, l) \in \mathbb{Z}_*^4 \mid \text{there are right } l \text{ components not in } \{n_1, n_2\}\},$$

for  $l = 0, 1, 2$  and

$$\Delta_3 = \{(i, j, k, l) \in \mathbb{Z}_*^4 \mid \text{there are at least 3 components not in } \{n_1, n_2\}\}.$$

**Lemma 3.4.** For fixed  $n_1, n_2$ , denote  $N = \max\{|n_1|, |n_2|\}$ . Let  $(i, j, k, l) \in (\Delta_0 \setminus \mathcal{N}) \cup \Delta_1 \cup (\Delta_2 \setminus \mathcal{N}) := \Delta$ , i.e there are at least 2 components in  $\{n_1, n_2\}$ , if

$$i+j-k-l=0,$$

then

$$|\lambda_i + \lambda_j - \lambda_k - \lambda_l| = |i^2 + j^2 - k^2 - l^2| \geq \frac{|j|}{N}.$$

*Proof.* As  $i + j - k - l = 0$ , then by direct calculation, we obtain

$$i^2 + j^2 - k^2 - l^2 = 2(j - k)(j - l).$$

Observe that  $j \neq k, l$ . Hence, if  $|j| \leq 2N$ , then

$$|2(j - k)(j - l)| \geq 2 \geq \frac{|j|}{N};$$

if  $|j| > 2N$ , at least one of  $k, l$  being in  $\{n_1, n_2\}$ , then

$$|2(j - k)(j - l)| \geq 2(|j| - N) \geq \frac{|j|}{N}.$$

□

**Lemma 3.5.** *Given  $\{n_1, n_2\} \in \mathcal{J}$ , there exists a real analytic, symplectic change of coordinates  $\Gamma$  in a neighbourhood of the origin in  $\ell^{a,p}$  which transforms hamiltonian  $H = \Lambda + G$  into Birkhoff normal form up to order four. That is*

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K,$$

where  $X_{\bar{G}}$ ,  $X_{\hat{G}}$  and  $X_K$  are real analytic vector fields from a neighbourhood of origin in  $\ell^{a,p}$  to  $\ell^{a,p-1}$ ,

$$\bar{G} = \frac{1}{4\pi} \sum_{i,j \in \mathbb{Z}_*} j |q_i|^2 |q_j|^2,$$

and

$$\|\hat{G}\|_{a,p-1} = O(\|q\|_{a,p}^4), \quad \|K\|_{a,p-1} = O(\|q\|_{a,p}^6).$$

Moreover,  $K(q, \bar{q}) \in \mathcal{A}_{n_1, n_2}$ .

*Proof.* Define

$$F = \frac{1}{2} \sum_{i+j-k-l=0} F_{ijkl} q_i q_j \bar{q}_k \bar{q}_l$$

with coefficients

$$iF_{ijkl} = \begin{cases} \frac{-jG_{ijkl}}{\lambda_i + \lambda_j - \lambda_k - \lambda_l}, & (i, j, k, l) \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \{\Lambda, F\} + G &= \frac{1}{2} \sum_{i+j-k-l=0} (jG_{ijkl} + i(\lambda_i + \lambda_j - \lambda_k - \lambda_l)F_{ijkl}) q_i q_j \bar{q}_k \bar{q}_l \\ &= \frac{1}{2} \sum_{\substack{i+j-k-l=0 \\ (i,j,k,l) \in (\Delta_0 \cap \mathcal{N}) \cup (\Delta_2 \cap \mathcal{N})}} jG_{ijkl} q_i q_j \bar{q}_k \bar{q}_l + \frac{1}{2} \sum_{\substack{i+j-k-l=0 \\ (i,j,k,l) \in \Delta_3}} jG_{ijkl} q_i q_j \bar{q}_k \bar{q}_l \\ &= \frac{1}{4\pi} \sum_{i,j \in \mathbb{Z}_*} j |q_i|^2 |q_j|^2 + \hat{G} \\ &= \bar{G} + \hat{G}, \end{aligned}$$

where  $\{\cdot, \cdot\}$  is a Poisson bracket with respect to the symplectic structure  $-i \sum_{j \in \mathbb{Z}_*} dq_j \wedge d\bar{q}_j$ .

Letting  $\Gamma = X_F^1$ , then

$$\begin{aligned} H \circ \Gamma &= H \circ X_F^t|_{t=1} \\ &= H + \{H, F\} + \int_0^1 (1-t) \{\{H, F\}, F\} \circ X_F^t dt \\ &= \Lambda + \{\Lambda, F\} + G + \{G, F\} + \int_0^1 (1-t) \{\{H, F\}, F\} \circ X_F^t dt \\ &= \Lambda + \bar{G} + \hat{G} + K, \end{aligned}$$



where

$$K = \{G, F\} + \frac{1}{2!} \{\{\Lambda, F\}, F\} + \frac{1}{2!} \{\{G, F\}, F\} \\ + \cdots + \frac{1}{n!} \{\cdots \{\Lambda, \underbrace{F, \dots, F}_n\}, F\} + \frac{1}{n!} \{\cdots \{G, \underbrace{F, \dots, F}_n\}, F\} + \cdots.$$

Now we prove the analyticity of the preceding transformation  $\Gamma$ . First, note that when  $(i, j, k, l) \in \Delta$ , we have  $|\lambda_i + \lambda_j - \lambda_k - \lambda_l| \geq \frac{|j|}{N}$ . So we know

$$\left| \frac{\partial F}{\partial \bar{q}_l} \right| \leq \sum_{i+j-k=l} \left| \frac{j G_{ijkl}}{\lambda_i + \lambda_j - \lambda_k - \lambda_l} \right| |q_i q_j \bar{q}_k| \leq c \sum_{i+j-k=l} |q_i q_j \bar{q}_k| = c(q * q * \bar{q})_l.$$

Hence, by Lemma 3.2,

$$\|F_{\bar{q}}\|_{a,p} \leq c \|q * q * \bar{q}\|_{a,p} \leq c \|q\|_{a,p}^3.$$

The analyticity of  $F_{\bar{q}}$  then follows from that of each of its component and its local boundedness. Moreover, it is clear that  $\|K\|_{a,p-1} \leq c \|q\|_{a,q}^6$ . The analogue claims for  $X_{\bar{G}}$  and  $X_{\hat{G}}$  are obvious.

We note that  $G$  and  $F$  have compact forms. Hence, by Lemma 2.2,  $\{G, F\}$  has a compact form. Since  $\Lambda$  is already in a compact form, repeating applications of Lemma 2.2 show that all terms of  $K$  have compact forms, so does  $K$ . Similarly, using Lemma 2.3, we can get that  $K$  has the gauge invariant property. Hence  $K \in \mathcal{A}_{n_1, n_2}$ .  $\square$

Now our Hamiltonian is  $\tilde{H} = \Lambda + \bar{G} + \hat{G} + K$ . Introduce the symplectic polar and complex coordinates by setting

$$\begin{cases} q_{n_j} = \sqrt{I_j + \xi_j} e^{i\theta_j}, & j = 1, 2; \\ q_j = z_j, & j \in \mathbb{Z}_1 = \mathbb{Z}_* \setminus \{n_1, n_2\}, \end{cases}$$

where  $n_1 \neq n_2$ ,  $\xi = \{\xi_1, \xi_2\} \in \mathbb{R}_+^2$ . Then

$$\Lambda = \sum_{1 \leq j \leq 2} \lambda_{n_j} (I_j + \xi_j) + \sum_{j \in \mathbb{Z}_1} \lambda_j z_j \bar{z}_j, \\ 4\pi \bar{G} = \sum_{1 \leq i, j \leq 2} n_j (I_i + \xi_i) (I_j + \xi_j) + \sum_{1 \leq i \leq 2, j \in \mathbb{Z}_1} j (I_i + \xi_i) z_j \bar{z}_j \\ + \sum_{1 \leq j \leq 2, i \in \mathbb{Z}_1} n_j (I_j + \xi_j) z_i \bar{z}_i + \sum_{i, j \in \mathbb{Z}_1} j z_i \bar{z}_i z_j \bar{z}_j,$$

where  $(\theta, I) \in \mathbb{T}^2 \times \mathbb{R}^2$  be standard angle-action variables in the  $(q_{n_1}, q_{n_2}, \bar{q}_{n_1}, \bar{q}_{n_2})$  space around  $\xi$ . Then we get

$$-i \sum_{j \in \mathbb{Z}_*} dq_j \wedge d\bar{q}_j = \sum_{1 \leq j \leq 2} d\theta_j \wedge dI_j - i \sum_{j \in \mathbb{Z}_1} dz_j \wedge d\bar{z}_j,$$

and Poisson bracket

$$\{F, G\} = \sum_{1 \leq j \leq 2} \frac{\partial F}{\partial \theta_j} \frac{\partial G}{\partial I_j} - \frac{\partial F}{\partial I_j} \frac{\partial G}{\partial \theta_j} - i \sum_{j \in \mathbb{Z}_1} \frac{\partial F}{\partial z_j} \frac{\partial G}{\partial \bar{z}_j} - \frac{\partial F}{\partial \bar{z}_j} \frac{\partial G}{\partial z_j}.$$

The new Hamiltonian, still denoted by  $\tilde{H}$ , up to a constant depending only on  $\xi$ , is given by

$$\tilde{H} = N + P = \langle \tilde{\omega}(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \tilde{\Omega}_j(\xi) z_j \bar{z}_j + \tilde{P}(I, \theta, z, \bar{z}, \xi),$$

where  $\tilde{\omega}(\xi) = (\tilde{\omega}_1(\xi), \tilde{\omega}_2(\xi))$  with

$$\begin{aligned}\tilde{\omega}_1(\xi) &= \lambda_{n_1} + \frac{1}{4\pi}((n_1 + n_1)\xi_1 + (n_1 + n_2)\xi_2), \\ \tilde{\omega}_2(\xi) &= \lambda_{n_2} + \frac{1}{4\pi}((n_2 + n_1)\xi_1 + (n_2 + n_2)\xi_2), \\ \tilde{\Omega}_j(\xi) &= \lambda_j + \frac{1}{4\pi}((n_1 + j)\xi_1 + (n_2 + j)\xi_2), \quad j \in \mathbb{Z}_1.\end{aligned}$$

At the same time, the perturbation is

$$\begin{aligned}\tilde{P} = & K + O(|I|^2) + O(|I|^2|\xi|) + O(|I| \sum_{j \in \mathbb{Z}_1} |j||z_j|^2) + O(|I| \sum_{j \in \mathbb{Z}_1} |z_j|^2) + O(|I||\xi| \sum_{j \in \mathbb{Z}_1} |z_j|^2) \\ & + O(\sum_{i,j \in \mathbb{Z}_1} |z_i|^2 |j||z_j|^2) + O(|\xi|^{\frac{1}{2}} \sum_{i=1}^2 \sum_{i+j-k=n_i} |z_i||jz_j||\bar{z}_k|) + O(\sum_{i+j-k-l=0} |z_i||jz_j||\bar{z}_k||\bar{z}_l|).\end{aligned}\tag{3.7}$$

**Lemma 3.6.** *If  $F(q, \bar{q}) \in \mathcal{A}_{n_1, n_2}$ , then after above symplectic polar and complex coordinates transform,  $F$  is still in  $\mathcal{A}_{n_1, n_2}$ .*

*Proof.* Suppose  $F_{k, \alpha, \beta} \neq 0$ , when  $\sum_n (\alpha_n - \beta_n)n = 0$  and  $\sum_n (\alpha_n - \beta_n) = 0$ . Without of the loss of generality, we consider term

$$F_{n_1, j, k, l} q_{n_1} q_j \bar{q}_k \bar{q}_l, \quad \text{with } n_1 + j - k - l = 0, \quad j, k, l \in \mathbb{Z}_1.$$

By the symplectic polar and complex coordinates transform, it becomes

$$F_{n_1, j, k, l} \sqrt{I_1 + \xi_1} e^{i\theta_1} z_j \bar{z}_k \bar{z}_l.$$

Obviously, it satisfies

$$\begin{aligned}k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n)n &= n_1 + j - k - l = 0, \\ k_1 + k_2 + \sum_n (\alpha_n - \beta_n) &= 1 + 1 - 1 - 1 = 0.\end{aligned}$$

The argument of the others terms is analogous to above and we omit it, then we complete the lemma.  $\square$

Now, let  $\varepsilon > 0$  be sufficiently small. Rescaling  $\xi_j$  by  $\varepsilon^4 \xi_j, j = 1, 2$ ,  $z, \bar{z}$  by  $\varepsilon^3 z, \varepsilon^3 \bar{z}$ , and  $I$  by  $\varepsilon^6 I$ , one obtains the rescaled Hamiltonian

$$\begin{aligned}H(I, \theta, z, \bar{z}, \xi) &= \varepsilon^{-10} \tilde{H}(\varepsilon^6 I, \theta, \varepsilon^3 z, \varepsilon^3 \bar{z}, \varepsilon^4 \xi) \\ &= \langle \omega^*(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j^*(\xi) z_j \bar{z}_j + \varepsilon P^*(I, \theta, z, \bar{z}, \xi),\end{aligned}\tag{3.8}$$

where  $\omega^*(\xi) = (\omega_1^*(\xi), \omega_2^*(\xi))$  with

$$\omega_1^*(\xi) = \varepsilon^{-4} \lambda_{n_1} + \frac{1}{4\pi}((n_1 + n_1)\xi_1 + (n_1 + n_2)\xi_2),\tag{3.9}$$

$$\omega_2^*(\xi) = \varepsilon^{-4} \lambda_{n_2} + \frac{1}{4\pi}((n_2 + n_1)\xi_1 + (n_2 + n_2)\xi_2),\tag{3.10}$$

$$\Omega_j^*(\xi) = \varepsilon^{-4} \lambda_j + \frac{1}{2\pi}((n_1 + j)\xi_1 + (n_2 + j)\xi_2), \quad j \in \mathbb{Z}_1.\tag{3.11}$$

The perturbation is

$$P^*(I, \theta, z, \bar{z}, \xi) = \varepsilon^{-11} \tilde{P}(\varepsilon^6 I, \theta, \varepsilon^3 z, \varepsilon^3 \bar{z}, \varepsilon^4 \xi). \quad (3.12)$$

Note that Eq. (1.1) has a conservation  $\int_0^{2\pi} |u|^2 dx = \sum_{j \neq 0} |q_j|^2 = c$ , i.e.

$$|q_{n_1}|^2 + |q_{n_2}|^2 + \sum_{j \in \mathbb{Z}_1} |q_j|^2 = c.$$

The above rescaling yields that

$$\varepsilon^2(I_1 + I_2) + (\xi_1 + \xi_2) + \varepsilon^2 \sum_{j \in \mathbb{Z}_1} |z_j|^2 = c,$$

that is

$$\xi_1 + \xi_2 = c - \varepsilon^2(I_1 + I_2 + \sum_{j \in \mathbb{Z}_1} |z_j|^2) = c + O(\varepsilon^2).$$

Let  $\omega(\xi) = (\omega_1(\xi), \omega_2(\xi))$ ,  $\Omega = (\Omega_j)_{j \in \mathbb{Z}_1}$ , where

$$\omega_1(\xi) = \varepsilon^{-4} n_1^2 + \frac{1}{4\pi} (n_1 + n_2) c + \frac{1}{4\pi} (n_1 - n_2) \xi_1, \quad (3.13)$$

$$\omega_2(\xi) = \varepsilon^{-4} n_2^2 + \frac{1}{4\pi} (n_1 + n_2) c + \frac{1}{4\pi} (n_2 - n_1) \xi_2, \quad (3.14)$$

$$\Omega_j(\xi) = \varepsilon^{-4} j^2 + \frac{1}{4\pi} (cj + n_1 \xi_1 + n_2 \xi_2). \quad (3.15)$$

We can write (3.8) as

$$H(I, \theta, z, \bar{z}, \xi) = \langle \omega(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j(\xi) z_j \bar{z}_j + P(I, \theta, z, \bar{z}, \xi), \quad (3.16)$$

where

$$P = \varepsilon P^* - \varepsilon^2 \frac{1}{2\pi} (n_1 + n_2) (I_1 + I_2 + \sum_{j \in \mathbb{Z}_1} |z_j|^2)^2. \quad (3.17)$$

## 4 Some Conditions

Consider the phase space

$$\mathcal{P}^{a,p} = \mathbb{T}^2 \times \mathbb{R}^2 \times \ell^{a,p} \times \ell^{a,p}$$

with the coordinates  $(\theta, I, z, \bar{z})$ . We denote  $\mathcal{P}_{\mathbb{C}}^{a,p}$  and  $\ell_{\mathbb{C}}^{a,p}$  the complexification of phase space  $\mathcal{P}^{a,p}$  and  $\ell^{a,p}$  respectively. Define a neighborhood of  $\mathbb{T}_0^2 = \mathbb{T}^2 \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$  by

$$\begin{aligned} D(s, r) &= \{(\theta, I, z, \bar{z}) : |\operatorname{Im} \theta| < s, |I| < r^2, \|z\|_{a,p} < r, \|\bar{z}\|_{a,p} < r\} \\ &\subset \mathbb{C}^2 \times \mathbb{C}^2 \times \ell_{\mathbb{C}}^{a,p} \times \ell_{\mathbb{C}}^{a,p} = \mathcal{P}_{\mathbb{C}}^{a,p}, \end{aligned}$$

where  $|\cdot|$  denotes the sup-norm for complex vectors,

Let  $\mathcal{O}$  be a neighborhood of the origin in  $\mathbb{R}_+^2$ . Define the difference operator  $\Delta_{\xi\zeta}$  in the variable  $\xi, \zeta \in \mathcal{O}$

$$\Delta_{\xi\zeta} = f(\cdot, \xi) - f(\cdot, \zeta).$$

We define the distance

$$|\Omega - \Omega'|_{-\delta, \mathcal{O}} = \sup_{\xi \in \mathcal{O}} \sup_{j \in \mathbb{Z}_1} j^{-\delta} |\Omega_j(\xi) - \Omega'_j(\xi)|,$$

and the Lipschitz semi-norm of frequencies  $\omega$  and  $\Omega$ :

$$|\omega|_{\mathcal{O}}^{\text{lip}} = \sup_{\substack{\xi, \zeta \in \mathcal{O}, \\ \xi \neq \zeta}} \frac{|\Delta_{\xi, \zeta} \omega|}{|\xi - \zeta|}, \quad |\Omega|_{-\delta, \mathcal{O}}^{\text{lip}} = \sup_{\substack{\xi, \zeta \in \mathcal{O}, \\ \xi \neq \zeta}} \sup_{j \in \mathbb{Z}_1} \frac{j^{-\delta} |\Delta_{\xi, \zeta} \Omega_j|}{|\xi - \zeta|},$$

for any real number  $\delta$ . Denote  $M = |\omega|_{\mathcal{O}}^{\text{lip}} + |\Omega|_{-\delta, \mathcal{O}}^{\text{lip}}$ .

For  $l = (l_1, \dots, l_n) \in \mathbb{Z}^n$ , we define

$$|l| = \sum_{j=1}^n |l_j|, \quad |l|_{\delta} = \sum_{j \neq 0} |j|^{\delta} |l_j|, \quad \langle l \rangle_{\delta} = \max\{1, |\sum_{j \neq 0} |j| l_j| \cdot |\sum_{j \neq 0} |j|^{\delta} |l_j|\},$$

and set

$$\Pi = \{(k, l) \neq 0, |l| \leq 2\} \subset \mathbb{Z}^2 \times \mathbb{Z}^{\infty}.$$

We now prove some propositions of the Hamiltonian in the normal form (3.8).

**Proposition 4.1.** *The map  $\xi \mapsto \omega(\xi)$  is a homeomorphism from  $\mathcal{O}$  to its image, which is Lipschitz continuous and its inverse also. The functions*

$$\xi \longrightarrow \frac{\Omega_j(\xi)}{|j|}$$

*are uniformly Lipschitz on  $\mathcal{O}$  for  $j \neq 0$ . Moreover, there exists a constant  $m > 0$  such that for all  $\xi \in \mathcal{O}$ ,*

$$|\langle l, \Omega(\xi) \rangle| \geq m \langle l \rangle_1, \quad \forall 1 \leq |l| \leq 2. \quad (4.1)$$

*Proof.* Rewrite  $\omega(\xi), \Omega(\xi)$  as  $\omega(\xi) = \alpha + A\xi$ ,  $\Omega(\xi) = \beta + B\xi$ , where

$$\alpha = \begin{pmatrix} \varepsilon^{-4} n_1^2 + \frac{1}{4\pi} (n_1 + n_2) c \\ \varepsilon^{-4} n_2^2 + \frac{1}{4\pi} (n_1 + n_2) c \end{pmatrix}, \quad A = \frac{1}{4\pi} \begin{pmatrix} n_1 - n_2 & 0 \\ 0 & n_2 - n_1 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} \vdots \\ \varepsilon^{-4} j^2 + \frac{1}{4\pi} c j \\ \vdots \end{pmatrix}_{j \in \mathbb{Z}_1}, \quad B = \frac{1}{4\pi} \begin{pmatrix} \vdots & \vdots \\ n_1 & n_2 \\ \vdots & \vdots \end{pmatrix}.$$

Since  $\det A = -\frac{1}{4\pi} (n_1 - n_2)^2 \neq 0$ , we have that  $\langle k, \omega(\xi) \rangle \neq 0$  for  $k \neq 0$  and the map  $\xi \mapsto \omega(\xi)$  is a Lipschitz homeomorphism.

For  $j \in \mathbb{Z}_1$

$$\left| \frac{\Omega_j(\xi)}{|j|} - \frac{\Omega_j(\zeta)}{|j|} \right| = \frac{1}{4\pi} \left| \frac{n_1(\xi_1 - \zeta_1)}{|j|} + \frac{n_2(\xi_2 - \zeta_2)}{|j|} \right| \leq \frac{\max\{|n_1|, |n_2|\}}{4\pi} |\xi - \zeta|.$$

At last, we prove inequality (4.1) in three cases:

Case I:  $|l| = 1$ , suppose  $l_j = 1, l_i = 0$ , for  $i \neq j$ , then

$$\begin{aligned} |\langle l, \Omega(\xi) \rangle| &= |\Omega_j(\xi)| = |\varepsilon^{-4} j^2 + \frac{1}{4\pi} (c j + n_1 \xi_1 + n_2 \xi_2)| \\ &\geq |\varepsilon^{-4} - \frac{1}{4\pi} c| |j|^2. \end{aligned}$$

Case II:  $|l| = 2$  and  $l_i = 1, l_j = 1$ , for  $i \neq j$ , then

$$\begin{aligned} |\langle l, \Omega(\xi) \rangle| &= |\Omega_i(\xi) + \Omega_j(\xi)| \geq (\varepsilon^{-4} - \frac{1}{4\pi} c) (|i|^2 + |j|^2) \\ &\geq \frac{1}{4} |\varepsilon^{-4} - \frac{1}{4\pi} c| (|i| + |j|)^2. \end{aligned}$$

Case III:  $|l| = 2$  and  $l_i = 1, l_j = -1$ , for  $i \neq j$ , then

$$\begin{aligned} |\langle l, \Omega(\xi) \rangle| &= |\Omega_i(\xi) - \Omega_j(\xi)| = |\varepsilon^{-4}(i^2 - j^2) + \frac{1}{4\pi}c(i - j)| \\ &\geq |\varepsilon^{-4} - \frac{1}{4\pi}c||i|^2 - |j|^2|. \end{aligned}$$

for properly selected  $m > 0$  we can show

$$|\langle l, \Omega(\xi) \rangle| \geq m\langle l \rangle_1.$$

Thus we prove this proposition.  $\square$

Setting  $\mathcal{O}_0$  consisting of all  $\xi \in \mathcal{O}$  such that

$$\begin{aligned} |\langle k, \omega(\xi) \rangle| &\geq \frac{\gamma}{|k|^\tau}, & k \neq 0, \\ |\langle k, \omega(\xi) \rangle \pm \Omega_j(\xi)| &\geq \frac{\gamma|j|^{1+\delta}}{|k|^\tau}, & k \neq 0, \\ |\langle k, \omega(\xi) \rangle \pm \Omega_i(\xi) \pm \Omega_j(\xi)| &\geq \frac{\gamma(|i| + |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, & k \neq 0, \\ |\langle k, \omega(\xi) \rangle + \Omega_i(\xi) - \Omega_j(\xi)| &\geq \frac{\gamma(|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, & k \neq 0 \text{ and } |i| \neq |j|, \\ |\langle k, \omega(\xi) \rangle + \Omega_j(\xi) - \Omega_{-j}(\xi)| &\geq \frac{\gamma|j|^\delta}{|k|^\tau}, & k \neq 0. \end{aligned}$$

**Proposition 4.2.**  $\text{meas}(\mathcal{O} \setminus \mathcal{O}_0) = O(\gamma)$ .

*Proof.* We consider the following nonresonant conditions:

$$\begin{aligned} \langle k, \omega(\xi) \rangle &\neq 0, \quad \forall k \in \mathbb{Z}^2, \\ \langle k, \omega(\xi) \rangle + \Omega_j(\xi) &\neq 0, \quad \forall k \in \mathbb{Z}^2, \\ \langle k, \omega(\xi) \rangle + \Omega_i(\xi) + \Omega_j(\xi) &\neq 0, \quad \forall k \in \mathbb{Z}^2, \\ \langle k, \omega(\xi) \rangle + \Omega_i(\xi) - \Omega_j(\xi) &\neq 0, \quad |k| + ||i| - |j|| \neq 0. \end{aligned}$$

As we have prove the first nonresonant condition in Proposition 4.1, we only consider the remaining three conditions. One has to check that  $\langle \alpha, k \rangle + \langle \beta, l \rangle \neq 0$  or  $Ak + B^T l \neq 0$  for  $1 \leq |l| \leq 2$ . Suppose  $Ak + B^T l = 0$ , for some  $k \in \mathbb{Z}^2$  and  $1 \leq |l| \leq 2$ . We let  $d$  be the sum of at most two nonzero components of  $l$ . Then

$$k_1(n_1 - n_2) + n_1 d = 0, \quad k_2(n_2 - n_1) + n_2 d = 0,$$

that is

$$k_1 = \frac{n_1}{n_2 - n_1}d, \quad k_2 = -\frac{n_2}{n_2 - n_1}d.$$

As  $n_1$  is odd and  $|n_2 - n_1| = 4$ , we have the following integer solutions

$$d = 0, \quad k_1 = k_2 = 0.$$

In this case,  $k = 0$ , “ $l$ ” have one “1” and one “-1”. Because the perturbation admit compact form with respect to  $n_1, n_2$ , that is

$$k_1 n_1 + k_2 n_1 + i l_i + j l_j = 0,$$

then we get  $i = j$ . It contradicts with our assumption  $i \neq j$ . Thus, we prove all nonresonant conditions.

The desired measure estimate of  $\text{meas}(\mathcal{O} \setminus \mathcal{O}_0)$  then follows from the same argument as that in Section 5.5.  $\square$

The next proposition is concerned with the Hamiltonian vector fields. The perturbation term  $P$  is real analytic in the space coordinates and Lipschitz in the parameters. Moreover, near  $\mathbb{T}_0^2$ , for each  $\xi \in \mathcal{O}$  its Hamiltonian vector field  $X_P = (P_I, -P_\theta, -iP_{\bar{z}}, iP_z)^T$  defines a real analytic map

$$X_P : D(s, r) \times \mathcal{O} \longrightarrow \mathcal{P}^{a, q}.$$

For  $s, r > 0$ , we introduce weighted norm for  $W = (X, Y, U, V) \in \mathcal{P}_{\mathbb{C}}^{a, q}$ ,

$$\|W\|_{r, q} = |X| + \frac{1}{r^2}|Y| + \frac{1}{r}\|U\|_{a, q} + \frac{1}{r}\|V\|_{a, q}.$$

Furthermore, for a map  $W : D(s, r) \times \mathcal{O} \longrightarrow \mathcal{P}_{\mathbb{C}}^{a, q}$ , for example, the Hamiltonian vector field  $X_P$ , we define the norms

$$\begin{aligned} \|W\|_{r, q, D(s, r) \times \mathcal{O}}^{\sup} &= \sup_{D(s, r) \times \mathcal{O}} \|W\|_{r, q}, \\ \|W\|_{r, q, D(s, r) \times \mathcal{O}}^{\text{lip}} &= \sup_{\substack{\xi, \zeta \in \mathcal{O}, \\ \xi \neq \zeta}} \sup_{D(s, r)} \frac{\|\Delta_{\xi, \zeta} W\|_{r, q}}{|\xi - \zeta|}. \end{aligned}$$

**Proposition 4.3.** (*Regularity of perturbation*) *There exists a neighborhood  $D(s, r)$  of  $\mathbb{T}_0^2$  in  $\mathcal{P}_{\mathbb{C}}^{a, p}$  such that  $P$  is defined on  $D(s, r) \times \mathcal{O}$ , and its Hamiltonian vector field defines a map*

$$X_P : D(s, r) \times \mathcal{O} \longrightarrow \mathcal{P}^{a, p-1},$$

*Moreover,  $X_P(\cdot, \xi)$  is real analytic on  $D(s, r)$  for each  $\xi \in \mathcal{O}$ , and  $X_P(\omega, \cdot)$  is uniformly Lipschitz on  $\mathcal{O}$  for each  $\omega \in D(s, r)$ .*

*Proof.* We first show that  $P_z \in \ell^{a, p-1}$ . From (3.12), it is obvious that  $\|P_z\|_{a, p-1} \leq c\varepsilon\|z\|_{a, p}$ . The other components of  $X_P$  is similar, and we get  $X_P \in \ell^{a, p-1}$ . Because of the form of  $\tilde{P}$  in (3.7) and  $P$  in (3.12), we know that  $\|P_z(\omega, \cdot)\|_{a, p-1}^{\text{lip}} \leq c\varepsilon\|z\|_{a, p}$ .  $X_P$  is Lipschitz continuous on  $\mathcal{O}$ , for all  $\omega \in D(s, r)$ .  $\square$

**Proposition 4.4.** (*The special form of the perturbation*) *The perturbation  $P$  in Hamiltonian (3.8) belongs to  $\mathcal{A}_{n_1, n_2}$ .*

*Proof.* Consider the Taylor-Fourier expansion of  $P$ :  $P = \sum_{k, \alpha, \beta} P_{k, \alpha, \beta}(I) e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta$ . It follows from  $K \in \mathcal{A}_{n_1, n_2}$  that  $P \in \mathcal{A}_{n_1, n_2}$ , i.e.

$$P_{k, \alpha, \beta}(I) = 0,$$

whenever

$$k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n) n \neq 0$$

or

$$k_1 + k_2 + \sum_n (\alpha_n - \beta_n) \neq 0.$$

This implies that  $P$  contains no terms of the form  $e^{i\langle k, \theta \rangle} z_j \bar{z}_{-j}$  with  $|j| > \frac{1}{2} \max\{|n_1|, |n_2|\} |k|$ . Especially,  $P$  contains no terms of the form  $z_j \bar{z}_{-j}$ . Together with Lemma 2.4, there is no terms of the form  $e^{i\langle k, \theta \rangle} z_n \bar{z}_n$  with  $k \neq 0$  and  $z_n \bar{z}_m$  with  $n \neq m$  in  $P$ .  $\square$

## 5 KAM step

To begin with the KAM iteration, we first fixed  $s, r > 0, p \geq \frac{3}{2}, d = 2, \delta = 1$  and restrict the Hamiltonian (3.8) on the domain  $D(s, r)$  and restrict the parameter to the set  $\mathcal{O}_0$ . Initially, we set  $\omega_0 = \omega$ ,  $\Omega_0 = \Omega$ ,  $P_0 = P$ ,  $r_0 = r$ ,  $s_0 = 0$  and  $M_0 = M$ . Consider the Hamiltonian :

$$N_0 = \langle \omega_0(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j^0(\xi) z_j \bar{z}_j, \quad H_0 = N_0 + P_0.$$

Hence,  $H_0$  is real analytic on  $D(s_0, r_0)$  and also depends on  $\xi \in \mathcal{O}_0$  Whitney smoothly. It is obviously that there is constant  $\varepsilon_0 > 0$  such that

$$\|X_P\|_{r, q, D(s_0, r_0) \times \mathcal{O}_0}^{\sup} + \frac{\gamma_0}{M_0} \|X_P\|_{r, q, D(s_0, r_0) \times \mathcal{O}_0}^{\text{lip}} \leq \varepsilon_0.$$

where  $\gamma_0 = \varepsilon_0^{\frac{1}{3}}$ .

We recall that

$$\begin{aligned} \mathcal{O}_0 &= \{\xi \in \mathcal{O} : |\langle k, \omega_0(\xi) \rangle| \geq \frac{\gamma_0}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_0(\xi) \rangle \pm \Omega_j^0(\xi)| &\geq \frac{\gamma_0 |j|^{1+\delta}}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_0(\xi) \rangle \pm \Omega_i^0(\xi) \pm \Omega_j^0(\xi)| &\geq \frac{\gamma_0 (|i| + |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_0(\xi) \rangle + \Omega_i^0(\xi) - \Omega_j^0(\xi)| &\geq \frac{\gamma_0 (|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, k \neq 0 \text{ and } |i| \neq |j|; \\ |\langle k, \omega_0(\xi) \rangle + \Omega_j^0(\xi) - \Omega_{-j}^0(\xi)| &\geq \frac{\gamma_0 |j|^\delta}{|k|^\tau}, k \neq 0. \} \end{aligned}$$

and  $P_0 = \sum_{k, \alpha, \beta} P_{k, \alpha, \beta}^0(I) e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta \in \mathcal{A}_{n_1, n_2}$ :  $k, \alpha, \beta$  have the following relations:

$$k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n) n = 0 \text{ and } k_1 + k_2 + \sum_n (\alpha_n - \beta_n) = 0.$$

Suppose that after a  $\nu$ -th KAM step, we arrive at a Hamiltonian

$$H = H_\nu = N_\nu + P_\nu, \quad N = N_\nu = \langle \omega_\nu(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j^\nu(\xi) z_j \bar{z}_j,$$

which is real analytic in  $(\theta, I, z, \bar{z}) \in D = D_\nu = D(s_\nu, r_\nu)$  for some  $s_\nu \leq s_0$ ,  $r_\nu \leq r_0$  and depends on  $\xi \in \mathcal{O}_\nu \subset \mathcal{O}_0$  Whitney smoothly, where

$$\begin{aligned} \mathcal{O}_\nu &= \{\xi : |\langle k, \omega_\nu(\xi) \rangle| \geq \frac{\gamma}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_\nu(\xi) \rangle \pm \Omega_j^\nu(\xi)| &\geq \frac{\gamma |j|^{1+\delta}}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_\nu(\xi) \rangle \pm \Omega_i^\nu(\xi) \pm \Omega_j^\nu(\xi)| &\geq \frac{\gamma (|i| + |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_\nu(\xi) \rangle + \Omega_i^\nu(\xi) - \Omega_j^\nu(\xi)| &\geq \frac{\gamma (|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, k \neq 0 \text{ and } |i| \neq |j|; \\ |\langle k, \omega_\nu(\xi) \rangle + \Omega_j^\nu(\xi) - \Omega_{-j}^\nu(\xi)| &\geq \frac{\gamma |j|^\delta}{|k|^\tau}, k \neq 0. \} \end{aligned}$$

for some  $\gamma_\nu \leq \gamma_0$ . We also assume that

$$\|X_P\|_{r,q,D_\nu(s_\nu,r_\nu) \times \mathcal{O}}^{\lambda_\nu} \leq \varepsilon_\nu.$$

for some  $\lambda_\nu = \frac{\gamma_\nu}{M_\nu}$  and  $0 < \varepsilon_\nu \leq \varepsilon_0$ .  $P = P_\nu = \sum_{k,\alpha,\beta} P_{k,\alpha,\beta}^0(I) e^{i\langle k,\theta \rangle} z^\alpha \bar{z}^\beta \in \mathcal{A}_{n_1,n_2}$ :  $k, \alpha, \beta$  have the following relations:

$$k_1 n_1 + k_2 n_2 + \sum_n (\alpha_n - \beta_n) n = 0 \text{ and } k_1 + k_2 + \sum_n (\alpha_n - \beta_n) = 0.$$

We will construct a symplectic transformation  $\Phi_\nu$ , such that

$$H_{\nu+1} = H_\nu \circ \Phi_\nu = N_{\nu+1} + P_{\nu+1}$$

with another normal form  $N_{\nu+1}$  and a smaller perturbation  $P_{\nu+1}$  which defined on a smaller domain  $D_{\nu+1}$ . We drop the index  $\nu$  of  $H_\nu, N_\nu, P_\nu, \Phi_\nu$  and shorten the index  $\nu+1$  as  $+$ . Also, throughout the whole paper, we use letters  $c, C$  to denote suitable (possibly different) constants that do not depend on the iteration steps.

## 5.1 The Homological Equations

Expand  $P$  into the Fourier-Taylor series

$$P = \sum_{k \in \mathbb{Z}^2, l \in \mathbb{N}^2, \alpha, \beta} P_{k,l,\alpha,\beta}(\xi) e^{i\langle k,\theta \rangle} I^l z^\alpha \bar{z}^\beta.$$

Let  $R$  be the truncation of  $P$  given by

$$R = \sum_{2|l|+|\alpha+\beta| \leq 2} \sum_{k \in \mathbb{Z}^2} P_{k,l,\alpha,\beta}(\xi) e^{i\langle k,\theta \rangle} I^l z^\alpha \bar{z}^\beta. \quad (5.1)$$

The mean value of such a Hamiltonian is defined as

$$[R] = \sum_{|l|+|\alpha|=1} P_{0,l,\alpha,\alpha}(\xi) I^l z^\alpha \bar{z}^\alpha$$

and is of the same form as  $N$ .

The coordinate transformation  $\Phi$  is obtained as the time-1-map  $X_F^t|_{t=1}$  of a Hamiltonian vector field  $X_F$ , where  $F$  is the same form as  $R$ . Using the Taylor formula we can write

$$\begin{aligned} H \circ \Phi &= N \circ X_F^1 + R \circ X_F^1 + (P - R) \circ X_F^1 \\ &= N + \{N, F\} + \int_0^1 (1-t) \{\{N, F\}, F\} \circ X_F^t dt \\ &\quad + R + \int_0^1 \{R, F\} \circ X_F^t dt + (P - R) \circ X_F^1 \\ &= N + R + \{N, F\} + \int_0^1 \{R + (1-t)\{N, F\}, F\} \circ X_F^t dt + (P - R) \circ X_F^1, \end{aligned}$$

where  $\{\cdot, \cdot\}$  is the Poisson bracket:

$$\{F, G\} = \sum_{1 \leq j \leq 2} \left( \frac{\partial F}{\partial \theta_j} \frac{\partial G}{\partial I_j} - \frac{\partial F}{\partial I_j} \frac{\partial G}{\partial \theta_j} \right) - i \sum_{j \in \mathbb{Z}_1} \left( \frac{\partial F}{\partial z_j} \frac{\partial G}{\partial \bar{z}_j} - \frac{\partial F}{\partial \bar{z}_j} \frac{\partial G}{\partial z_j} \right).$$



In view of the previous equation, we define the new normal form by  $N_+ = N + \widehat{N}$ , where  $\widehat{N}$  satisfies the so-called homological equation (the unknown are  $F$  and  $\widehat{N}$ ):

$$\{F, N\} + \widehat{N} = R. \quad (5.2)$$

Once the homological equation is solved, we define the new perturbation term  $P_+$  by

$$P_+ = \int_0^1 \{R(t), F\} \circ X_F^t dt + (P - R) \circ X_F^1,$$

where  $R(t) = tR + (1 - t)\widehat{N}$ .

**Lemma 5.1.** *The homological Equation (5.2) has a solution  $F, \widehat{N}$  which is unique with  $[F] = 0$ ,  $[\widehat{N}] = \widehat{N}$ ,  $F$  is regular on  $D(s, r) \times \mathcal{O}$  in the above sense, and satisfies for  $0 < \sigma < s$  the estimates*

$$\begin{aligned} \|X_F\|_{r,p,D(s-\sigma,r) \times \mathcal{O}}^{\sup} &\leq \frac{C}{\gamma\sigma^{2\tau+3}} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\sup}, \\ \|X_F\|_{r,p,D(s-\sigma,r) \times \mathcal{O}}^{\text{lip}} &\leq \frac{C}{\gamma\sigma^{2\tau+3}} \left( \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\text{lip}} + \frac{M}{\gamma} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\sup} \right), \end{aligned}$$

and

$$\begin{aligned} \|X_{\widehat{N}}\|_{r,q,D(s-\sigma,r) \times \mathcal{O}}^{\sup} &\leq C \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\sup}, \\ \|X_{\widehat{N}}\|_{r,q,D(s-\sigma,r) \times \mathcal{O}}^{\text{lip}} &\leq C \left( \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\text{lip}} + \frac{M}{\gamma} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\sup} \right). \end{aligned}$$

*Proof.* Decompose  $R = R^0 + R^1 + R^2$ , where  $R^j$  comprises all terms in the expansion of  $R$  with  $|\alpha + \beta| = j$ . Decompose similarly  $F, N$  and  $\widehat{N}$ , where necessarily  $N^1 = 0$  and  $\widehat{N}^1 = 0$  by normalization. The homological equation decomposes into

$$\begin{aligned} \{F^0, N\} + \widehat{N}^0 &= R^0, \\ \{F^1, N\} &= R^1, \\ \{F^2, N\} + \widehat{N}^2 &= R^2. \end{aligned} \quad (5.3)$$

We will see that with the chosen normalization and the Diophantine conditions these equations determine  $\widehat{N}^0, F^0, F^1$  and then  $\widehat{N}^2, F^2$  uniquely.

Due to independence of  $z, \bar{z}$ , the first equation amounts to the classical, finite-dimensional partial differential equation

$$\partial_\omega F^0 + \widehat{N}^0 = R^0, \quad \partial_\omega = \sum_{1 \leq i \leq 2} \omega_i \partial_{\theta_i}.$$

This leads to  $\widehat{N}^0 = [R^0]$  and  $\partial_\omega F^0 = R^0 - [R^0]$  with  $[F^0] = 0$ . Their estimates are standard and of the same form — indeed much better — than the ones for  $F^1, F^2$  and  $\widehat{N}^2$  obtained below. For later reference, we record that

$$\begin{aligned} \|X_{F^0}\|_{r,p,D(s-2\sigma,r) \times \mathcal{O}}^{\sup} &\leq \frac{C}{\gamma\sigma^{2\tau+3}} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\sup}, \\ \|X_{F^0}\|_{r,p,D(s-4\sigma,r) \times \mathcal{O}}^{\text{lip}} &\leq \frac{C}{\gamma\sigma^{2\tau+3}} \left( \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\text{lip}} + \frac{M}{\gamma} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^{\sup} \right). \end{aligned}$$

Note that  $X_{F^0}$  does not have any  $z, \bar{z}$  component, so  $\|X_{F^0}\|_{r,p}$  does not depend on  $p$ .

Consider the second equation in (5.3). Writing

$$R^1 = R^{10} + R^{01} = \langle \mathcal{R}^{10}, z \rangle + \langle \mathcal{R}^{01}, \bar{z} \rangle$$

and similarly  $F^1$  it decomposes into

$$\{F^{ij}, N\} = R^{ij}, \quad i + j = 1,$$

and it suffices to study each equation individually.

We have  $\mathcal{R}^{10} = R_z|_{z=\bar{z}=0}$  and thus

$$\frac{1}{r} \|\mathcal{R}^{10}\|_{a,q,D(s)}^{\sup} \leq \|X_R\|_{r,q,D(s,r)}^{\sup},$$

where  $D(s) = \{\theta : |\operatorname{Im} \theta| < s\}$ . Writing  $R^{10} = \langle R^{10}, z \rangle = \sum_{j \in \mathbb{Z}_1} R_j(\theta, \xi) z_j$ , and similarly  $F^{10}$ , the equation  $\{F^{10}, N\} = R^{10}$  further decomposes into

$$\partial_\omega F_j - i\Omega_j F_j = R_j, \quad j \in \mathbb{Z}_1.$$

Expanding  $R_j$  in a Fourier series,  $R_j = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{R}_{jk} e^{i\langle k, \theta \rangle}$ , and similarly  $F_j$ . Then above homological equation can be written in

$$i(\langle k, \omega \rangle - \Omega_j) \hat{F}_{jk} = \hat{R}_{jk}, \quad j \in \mathbb{Z}_1.$$

By the non-degeneracy condition (4.1) and Diophantine condition, we have uniformly on  $\mathcal{O}$

$$|\Omega_j(\xi)| \geq m|j|^d, \quad j \in \mathbb{Z}_1,$$

and

$$|\langle k, \omega(\xi) \rangle - \Omega_j(\xi)| \geq \frac{\gamma|j|^{1+\delta}}{|k|^\tau}, \quad k \neq 0 \text{ and } j \in \mathbb{Z}_1.$$

Using the estimates in Lemma A.3, the unique solution  $F_j$  satisfies the estimate

$$|F_j|_{D(s-2\sigma)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+1}|j|^{1+\delta}} |R_j|_{D(s-\sigma)}^{\sup}, \quad j \in \mathbb{Z}_1.$$

Since  $p - q \leq \delta$ , this and Lemma A.4 imply

$$\|\mathcal{F}^{10}\|_{a,p,D(s-2\sigma)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+1}} \|\mathcal{R}^{10}\|_{a,q,D(s)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+1}} r \|X_R\|_{r,q,D(s,r)}^{\sup}.$$

The same estimate holds for  $\mathcal{F}^{01}$ . Multiplying  $\mathcal{F}^{10}$  with  $z$  and  $\mathcal{F}^{01}$  with  $\bar{z}$  and using  $p > \frac{3}{2}$  this gives

$$\frac{1}{r^2} |F^1|_{D(s-2\sigma,r)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+1}} \|X_R\|_{r,q,D(s,r)}^{\sup},$$

finally with Cauchy's estimate

$$\|X_{F^1}\|_{r,a,p,D(s-3\sigma,r)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+1}} \|X_R\|_{r,q,D(s,r)}^{\sup}.$$

To obtain Lipschitz estimates, we study first the differences  $\Delta F_j = F_j(\xi) - F_j(\zeta)$  for  $\xi, \zeta \in \mathcal{O}$ . we obtain

$$\partial_\omega \Delta F_j - i\Omega_j \Delta F_j = \Delta R_j + \partial_{\Delta\omega} F_j + i\Delta\Omega_j F_j, \quad j \in \mathbb{Z}_1.$$

The right hand side is known, so  $\Delta F_j$  uniquely solves the same kind of equation as  $F_j$ . So we obtain

$$\begin{aligned} |\Delta F_j|_{D(s-3\sigma)}^{\sup} &\leq \frac{C}{\gamma\sigma^{\tau+1}|j|^{1+\delta}} \left( |\Delta R_j|_{D(s-\sigma)}^{\sup} + \frac{1}{\sigma} |F_j|_{D(s-2\sigma)}^{\sup} (|\Delta\omega| + |\Delta\Omega_j|_{D(s)}^{\sup}) \right) \\ &\leq \frac{C}{\gamma\sigma^{\tau+1}|j|^{1+\delta}} |\Delta R_j|_{D(s-\sigma)}^{\sup} + \frac{C}{\gamma^2\sigma^{2\tau+3}|j|^{2(1+\delta)}} |R_j|_{D(s-\sigma)}^{\sup} (|\Delta\omega| + |\Delta\Omega_j|_{D(s)}^{\sup}). \end{aligned}$$

Then

$$\|\Delta \mathcal{F}^{10}\|_{a,p,D(s-3\sigma)}^{\sup} \leq \frac{C}{\gamma \sigma^{\tau+1}} \|\Delta \mathcal{R}^{10}\|_{a,q,D(s)}^{\sup} + \frac{C}{\gamma^2 \sigma^{2\tau+3}} \|\mathcal{R}^{10}\|_{D(s)}^{\sup} (|\Delta \omega| + |\Delta \Omega|_{-\delta,D(s)}^{\sup}).$$

Dividing by  $|\xi - \zeta| \neq 0$  and taking the supremum over  $\mathcal{O}$ ,

$$\begin{aligned} \|\mathcal{F}^{10}\|_{a,p,D(s-3\sigma)}^{\text{lip}} &\leq \frac{C}{\gamma \sigma^{\tau+1}} \|\mathcal{R}^{10}\|_{a,q,D(s)}^{\text{lip}} + \frac{C}{\gamma^2 \sigma^{2\tau+3}} \|\mathcal{R}^{10}\|_{D(s)}^{\sup} (|\omega|_{\mathcal{O}}^{\text{lip}} + |\Omega|_{-\delta,\mathcal{O}}^{\text{lip}}) \\ &\leq \frac{C}{\gamma \sigma^{2\tau+3}} \left( \|\mathcal{R}^{10}\|_{a,q,D(s)}^{\text{lip}} + \frac{M}{\gamma} \|\mathcal{R}^{10}\|_{D(s)}^{\sup} \right). \end{aligned}$$

The same estimate applies to  $\mathcal{F}^{01}$ . So for the vector field of  $F^1$ , we finally get

$$\|X_{F^1}\|_{r,p,D(s-4\sigma)}^{\text{lip}} \leq \frac{C}{\gamma \sigma^{2\tau+3}} \left( \|X_R\|_{r,q,D(s,r)}^{\text{lip}} + \frac{M}{\gamma} \|X_R\|_{r,q,D(s,r)}^{\sup} \right).$$

This concludes the discussion of  $F^1$ .

Now we consider the third equation in (5.3). Write  $R^2 = R^{20} + R^{11} + R^{02}$  and similarly  $F^2$  and  $N^2$ . This equation decomposes into

$$\{F^{ij}, N\} + \hat{N}^{ij} = R^{ij}, \quad (5.4)$$

while  $\hat{N}^{ij} = 0$  for  $i \neq j$ .

Consider the equation for  $F^{11}$ , which is slightly more complicated than the ones for  $F^{20}$  and  $F^{02}$ . Writing  $R^{11} = \langle \mathcal{R}^{11} z, \bar{z} \rangle$ , we have  $\mathcal{R}^{11} = R_{z\bar{z}}|_{z=\bar{z}=0}$ . Thus,  $R^{11}$  is the Jacobian of  $R_z$  with respect to  $\bar{z}$  at  $\bar{z} = 0$ . By Cauchy's inequality, we have

$$\|\mathcal{R}^{11}\|_{q,p,D(s)}^{\sup} \leq \frac{1}{r} \|R_z\|_{q,D(s,r)}^{\sup} \leq \|X_R\|_{r,q,D(s,r)}^{\sup},$$

where  $\|\cdot\|_{q,p}$  denotes the operator norm included by  $\|\cdot\|_{a,p}$  and  $\|\cdot\|_{a,p}$  in the source and target spaces, respectively.

Note that  $P$  contains no terms of the form  $z_i \bar{z}_j$  with  $i \neq j$  and  $e^{i(k,\theta)} z_j \bar{z}_j$  with  $k \neq 0$ , write more explicitly

$$R^{11} = \sum_{\substack{|i| \neq |j| \\ i,j \in \mathbb{Z}_1}} R_{ij}(\theta, \xi) z_i \bar{z}_j + \sum_{j \in \mathbb{Z}_1} R_{jj}(\xi) z_j \bar{z}_j + \sum_{\substack{j \in \mathbb{Z}_1 \\ |j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\} |k|}} R_{j(-j)}(\theta, \xi) z_j \bar{z}_{-j},$$

and similarly  $F^{11}$ . The Eq. (5.4) decomposes into

$$\partial_\omega F_{ij} - i(\Omega_i - \Omega_j) F_{ij} = R_{ij}, \quad i \neq j.$$

and  $F_{jj} = 0$  for  $j \neq 0$ ,  $F_{j(-j)} = 0$  for  $|j| > \frac{1}{2} \max\{|n_1|, |n_2|\} |k|$ .

Again, by Diophantine condition, we have

$$|\langle k, \omega(\xi) \rangle + (\Omega_i(\xi) - \Omega_j(\xi))| \geq \frac{\gamma(|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, \quad k \neq 0 \text{ and } |i| \neq |j|,$$

and

$$|\langle k, \omega(\xi) \rangle + (\Omega_j(\xi) - \Omega_{-j}(\xi))| \geq \frac{\gamma|j|^\delta}{|k|^\tau}, \quad k \neq 0,$$

then we obtain

$$(|i|^\delta + |j|^\delta) |F_{ij}|_{D(s-2\sigma)}^{\sup} \leq \frac{C}{\gamma \sigma^{\tau+1} ||i| - |j||} |R_{ij}|_{D(s-\sigma,r)}^{\sup}, \quad |i| \neq |j|,$$

and

$$|j|^\delta |F_{j(-j)}|_{D(s-2\sigma)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+1}} |R_{j(-j)}|_{D(s-\sigma,r)}^{\sup}.$$

With Lemma A.4, this yield

$$\|\mathcal{F}^{11}\|_{p,p,D(s-2\sigma)}^{\sup}, \|\mathcal{F}^{11}\|_{q,q,D(s-2\sigma)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+3}} \|\mathcal{R}^{11}\|_{q,p,D(s)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+3}} \|X_R\|_{r,q,D(s,r)}^{\sup},$$

The same, and even better estimates hold for  $F^{20}$  and  $F^{02}$ . Multiplying with  $z, \bar{z}$  we then get

$$\frac{1}{r^2} |F^2|_{D(s-2\sigma,r)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+3}} \|X_R\|_{r,q,D(s,r)}^{\sup},$$

finally with Cauchy's estimate

$$\|X_{F^2}\|_{r,p,D(s-3\sigma,r)}^{\sup} \leq \frac{C}{\gamma\sigma^{\tau+3}} \|X_R\|_{r,q,D(s,r)}^{\sup}.$$

The estimate for the Lipschitz semi-norm of  $X_{F^2}$  is obtained by the same arguments as the one for  $X_{F^1}$ , and the result is analogous. We therefore omit it.

The estimates of  $X_{\hat{N}}$  follow from the observation that

$$\hat{N} = \sum_{|l|=1} P_{0l00} I^l + \sum_{j \in \mathbb{Z}_1} P_{00jj}(\xi) z_j \bar{z}_j.$$

The final estimates of the lemma are obtained by replacing  $\sigma$  by  $\frac{\sigma}{4}$  throughout the proof.  $\square$

For  $\lambda \geq 0$ , we define

$$\|X\|_r^\lambda = \|X\|_r^{\sup} + \lambda \|X\|_r^{\text{lip}}.$$

The symbol “ $\lambda$ ” in  $\|X\|_r^\lambda$  will always be used in this role and never has the meaning of exponentiation.

**Lemma 5.2.** *The estimates of Lemma 5.1 imply that*

$$\begin{aligned} \|X_F\|_{r,p,D(s-\sigma,r) \times \mathcal{O}}^\lambda &\leq \frac{C}{\gamma\sigma^{2\tau+3}} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda, \\ \|X_{\hat{N}}\|_{r,q,D(s-\sigma,r) \times \mathcal{O}}^\lambda &\leq C \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda, \end{aligned}$$

for  $0 < \sigma < s$  and  $0 \leq \lambda \leq \frac{\gamma}{M}$  with another  $C$  of the same form as in Lemma 5.1.

The preceding lemma also gives us an estimate of  $\|DX_F\|_{r,p,D(s-2\sigma,r) \times \mathcal{O}}^\lambda$  with the help of Cauchy's estimate.

**Lemma 5.3.** *Under the assumptions of Lemma 5.1,*

$$\|DX_F\|_{r,p,p,D(s-2\sigma,r) \times \mathcal{O}}^\lambda, \|DX_F\|_{r,q,q,D(s-2\sigma,r) \times \mathcal{O}}^\lambda \leq \frac{C}{\gamma\sigma^{2\tau+3}} \|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda.$$

*Proof.* The proof can be found on page 160 in [22].  $\square$

We recall some approximation results in [27], which show that the second order approximation of  $P$  can be controlled by  $P$ , and that  $P - R$  is small when we contract the domain (this contraction is governed by the new parameter  $\eta$ ).

**Lemma 5.4.** *Let  $P$  satisfies Proposition 4.3 and consider its Taylor approximation  $R$  of the form (5.1). Then there exists  $C > 0$  so that for all  $\eta > 0$*

$$\|X_R\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda \leq \|X_P\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda, \quad (5.5)$$

$$\|X_P - X_R\|_{\eta r,q,D(s,4\eta r) \times \mathcal{O}}^\lambda \leq C\eta \|X_P\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda. \quad (5.6)$$

At the end, we give some estimates for  $X_F^t$ . The formulas (5.7) and (5.8) will be used to prove our coordinate transformation is well-defined. Inequalities (5.9) and (5.10) will be used to check the convergence of the iteration.

**Lemma 5.5.** *If  $\|X_P\|_{r,q,D(s,r) \times \mathcal{O}}^\lambda \leq \frac{\gamma\sigma^{2\tau+4}\eta^2}{C}$ , we then have*

$$X_F^t : D(s - 2\sigma, \frac{r}{2}) \longrightarrow D(s - \sigma, r), \quad -1 \leq t \leq 1. \quad (5.7)$$

Similarly,

$$X_F^t : D(s - 3\sigma, \frac{r}{4}) \longrightarrow D(s - 2\sigma, \frac{r}{2}), \quad -1 \leq t \leq 1. \quad (5.8)$$

Moreover,

$$\|X_F^t - Id\|_{r,p,D(s-2\sigma,\frac{r}{2}) \times \mathcal{O}}^\lambda < C\|X_F\|_{r,p,D(s-\sigma,r) \times \mathcal{O}}^\lambda, \quad (5.9)$$

$$\|DX_F^t - Id\|_{r,q,q,D(s-3\sigma,\frac{r}{4}) \times \mathcal{O}}^\lambda < C\|DX_F\|_{r,q,q,D(s-\sigma,r) \times \mathcal{O}}^\lambda, \quad (5.10)$$

for  $0 \leq \lambda \leq \frac{\gamma}{M}$ . The latter estimate also holds in the  $\|\cdot\|_{r,p,p}$  - norm.

We can use Lemma 5.2 and Lemma A.5 to prove this lemma.

## 5.2 The New Hamiltonian

The map  $\Phi = X_F^1$  defined above transforms  $H$  into  $H \circ \Phi = N_+ + P_+$  on  $D(s - \sigma, \frac{r}{2})$ , where  $N_+ = N + \widehat{N}$  and

$$P_+ = \int_0^1 \{R(t), F\} \circ X_F^t dt + (P - R) \circ X_F^1,$$

where  $R(t) = tR + (1 - t)\widehat{N}$ . Hence

$$X_{P_+} = \int_0^1 (X_F^t)^* [X_{R(t)}, X_F] dt + (X_F^1)^* (X_P - X_R).$$

From the paper [27], we have known the following result:

$$\|(X_F^t)^* Y\|_{\eta r,q,D(s-4\sigma,\eta r)}^\lambda \leq C\|Y\|_{\eta r,q,D(s-2\sigma,4\eta r)}^\lambda, \quad 0 \leq t \leq 1. \quad (5.11)$$

We already have estimated  $\|X_P - X_R\|_{\eta r,q}^\lambda$  in (5.6), so it remains to consider the commutator  $\|[X_{R(t)}, X_F]\|_{r,q}$ .

First, we have

$$\begin{aligned} \|X_{R(t)}\|_{r,q,D(s-\sigma,r)}^\lambda &\leq \|X_{\widehat{N}}\|_{r,q,D(s-\sigma,r)}^\lambda + \|X_R\|_{r,q,D(s-\sigma,r)}^\lambda \\ &\leq C\|X_P\|_{r,q,D(s,r)}^\lambda. \end{aligned}$$

Moreover, we have the pointwise estimate

$$\begin{aligned} \|[X_{R(t)}, X_F]\|_{r,q} &\leq \|DX_{R(t)} \cdot X_F\|_{r,q} + \|X_{R(t)} \cdot DX_F\|_{r,q} \\ &\leq \|DX_{R(t)}\|_{r,q,p} \|X_F\|_{r,p} + \|DX_F\|_{r,q,q} \|X_{R(t)}\|_{r,q}. \end{aligned}$$

By the product rule for Lipschitz-norms and Cauchy's estimate, we thus obtain

$$\begin{aligned} \| [X_{R(t)}, X_F] \|_{r,q,D(s-2\sigma,\frac{r}{2})}^\lambda &\leq \| DX_{R(t)} \|_{r,q,p,D(s-2\sigma,\frac{r}{2})}^\lambda \| X_F \|_{r,p,D(s-2\sigma,\frac{r}{2})}^\lambda \\ &\quad + \| DX_F \|_{r,q,q,D(s-2\sigma,\frac{r}{2})}^\lambda \| X_{R(t)} \|_{r,q,D(s-2\sigma,\frac{r}{2})}^\lambda \\ &\leq \frac{C}{\gamma\sigma^{2\tau+1}} \left( \| X_P \|_{r,q,D(s,r)}^\lambda \right)^2, \end{aligned}$$

for  $0 \leq \lambda \leq \frac{\gamma}{M}$ . Hence, also

$$\begin{aligned} \| [X_{R(t)}, X_F] \|_{\eta r,q,D(s-2\sigma,\frac{r}{2})}^\lambda &\leq \frac{1}{\eta^2} \| [X_{R(t)}, X_F] \|_{r,q,D(s-2\sigma,\frac{r}{2})}^\lambda \\ &\leq \frac{C}{\gamma\sigma^{2\tau+3}\eta^2} \left( \| X_P \|_{r,q,D(s,r)}^\lambda \right)^2. \end{aligned}$$

Together with the estimate on  $\| X_P - X_R \|_{\eta r,q}^\lambda$  in (5.6) and with that in (5.11), we finally arrive at the estimate

$$\| X_{P_+} \|_{\eta r,q,D(s-2\sigma,\frac{r}{2})}^\lambda \leq C\eta \| X_P \|_{r,q,D(s,r)}^\lambda + \frac{C}{\gamma\sigma^{2\tau+3}\eta^2} \left( \| X_P \|_{r,q,D(s,r)}^\lambda \right)^2,$$

for  $0 \leq \lambda \leq \frac{\gamma}{M}$ . This is the bound for the new perturbation.

Now turn to the new frequencies  $\omega_+(\xi) = \omega(\xi) + \widehat{\omega}(\xi)$  and  $\Omega_+(\xi) = \Omega(\xi) + \widehat{\Omega}(\xi)$ . For  $\widehat{N}$ , we have the estimate

$$\| X_{\widehat{N}} \|_{r,q,D(s-\sigma,r) \times \mathcal{O}}^\lambda \leq C \| X_R \|_{r,q,D(s,r) \times \mathcal{O}}^\lambda,$$

for  $0 \leq \lambda \leq \frac{\gamma}{M}$ . The weighted norm implies that we have  $|\widehat{\omega}(\xi)| \leq \| X_{\widehat{N}} \|_{r,q}^{\sup}$  and  $\| \widehat{\Omega}(\xi) z \|_{a,q} \leq r \| X_{\widehat{N}} \|_{r,q}^{\sup}$  on  $D(s,r)$  and consequently  $|\widehat{\Omega}(\xi)|_{q-p} \leq \| X_{\widehat{N}} \|_{r,q}^{\sup}$ . The same holds for the Lipschitz semi-norms. Since  $p - q \leq \delta$ , we obtain

$$|\widehat{\omega}|_{\mathcal{O}}^\lambda + |\widehat{\Omega}|_{-\delta,\mathcal{O}}^\lambda \leq C \| X_P \|_{r,q,D(s,r) \times \mathcal{O}}^\lambda, \quad (5.12)$$

where  $\Omega = (\Omega_j)_{j \in \mathbb{Z}_1}$  and  $\widehat{\Omega} = (\widehat{\Omega}_j)_{j \in \mathbb{Z}_1}$ .

In order to control the assumptions of the KAM step for the iteration, we notice that the last estimate also implies

$$|\langle l, \widehat{\Omega}(\xi) \rangle| \leq \langle l \rangle_\delta |\widehat{\Omega}|_{-\delta} \leq \langle l \rangle_{d-1} \| X_P \|_{r,q,D(s,r) \times \mathcal{O}}^\lambda. \quad (5.13)$$

**Lemma 5.6.**  $P_+ \in \mathcal{A}_{n_1,n_2}$ .

*Proof.* Note that

$$\begin{aligned} P_+ = & P - R + \{P, F\} + \frac{1}{2!} \{ \{N, F\}, F \} + \frac{1}{2!} \{ \{P, F\}, F \} \\ & + \dots + \frac{1}{n!} \{ \dots \{N, \underbrace{F \dots F}_n, F \} + \frac{1}{n!} \{ \dots \{P, \underbrace{F \dots F}_n, F \} + \dots \}. \end{aligned}$$

Since  $P \in \mathcal{A}_{n_1,n_2}$ , then  $F$ , so do  $P - R$ ,  $\{N, F\}$  and  $\{P, F\}$ . The lemma follows from Lemma 2.2 and Lemma 2.3.  $\square$

### 5.3 Iteration Lemma

For given  $\varepsilon_0, m_0 = m, M_0 = M, r_0 = r, s_0 = s < 1$ . Moreover, we define sequences as follows:

$$\begin{aligned}\sigma_0 &= \frac{s}{8}, \quad \sigma_{\nu+1} = \frac{\sigma_\nu}{2}, \quad s_{\nu+1} = s_\nu - 2\sigma_\nu, \\ \eta_\nu^3 &= \frac{\varepsilon_\nu}{\gamma_\nu \sigma_\nu^{2\tau+3}}, \quad r_{\nu+1} = \eta_\nu r_\nu, \quad D_\nu = D(s_\nu, r_\nu), \\ \gamma_0 &= \varepsilon_0^{\frac{1}{3}}, \quad \gamma_\nu = \varepsilon_\nu^{\frac{1}{3}}, \quad M_\nu = M_0(2 - 2^\nu), \quad \lambda_\nu = \frac{\gamma_\nu}{M_\nu}, \\ m_\nu &= \frac{m_0}{2}(1 + 2^{-\nu}), \quad \varepsilon_{\nu+1} = C(\gamma_\nu \sigma_\nu^{2\tau+3})^{-\frac{1}{3}} \varepsilon_\nu^{\frac{4}{3}},\end{aligned}$$

$$\begin{aligned}\mathcal{O}_\nu &= \{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\xi) \rangle| \geq \frac{\gamma_\nu}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_\nu(\xi) \rangle \pm \Omega_j^\nu(\xi)| &\geq \frac{\gamma_\nu |j|^{1+\delta}}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_\nu(\xi) \rangle \pm \Omega_i^\nu(\xi) \pm \Omega_j^\nu(\xi)| &\geq \frac{\gamma_\nu(|i| + |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, k \neq 0; \\ |\langle k, \omega_\nu(\xi) \rangle + \Omega_i^\nu(\xi) - \Omega_j^\nu(\xi)| &\geq \frac{\gamma_\nu(|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, k \neq 0 \text{ and } |i| \neq |j|; \\ |\langle k, \omega_\nu(\xi) \rangle + \Omega_j^\nu(\xi) - \Omega_{-j}^\nu(\xi)| &\geq \frac{\gamma_\nu |j|^\delta}{|k|^\tau}, k \neq 0\}.\end{aligned}$$

The proceeding analysis can be summarized as follows:

**Lemma 5.7.** *Let*

$$\varepsilon_0 \leq \frac{\gamma_0 \sigma_0^{2\tau+6}}{C^3}, \quad \gamma_0 \leq \frac{m_0}{2}.$$

*Suppose,  $H_\nu = N_\nu + P_\nu$  is given on  $D(s_\nu, r_\nu) \times \mathcal{O}_\nu$  which is real analytic in  $(\theta, I, z, \bar{z}) \in D(s_\nu, r_\nu)$  and Whitney smooth in  $\xi \in \mathcal{O}_\nu$ , where*

$$N_\nu = \langle \omega_\nu(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j^\nu(\xi) z_j \bar{z}_j.$$

*Its coefficients satisfy*

$$\begin{aligned}|\omega_\nu|_{\mathcal{O}_\nu}^{\text{lip}} + |\Omega^\nu|_{-\delta, \mathcal{O}_\nu}^{\text{lip}} &\leq M_\nu, \\ |\omega_\nu - \omega_{\nu-1}|_{\mathcal{O}_\nu}^{\lambda_\nu} &\leq \varepsilon_{\nu-1}, \\ |\Omega^\nu - \Omega^{\nu-1}|_{-\delta, \mathcal{O}_\nu}^{\lambda_\nu} &\leq \varepsilon_{\nu-1},\end{aligned}$$

*and*

$$|\langle l, \Omega^\nu(\xi) \rangle| \geq m_\nu \langle l \rangle_{d-1}, \quad \forall 1 \leq |l| \leq 2. \quad (5.14)$$

*$P_\nu \in \mathcal{A}_{n_1, n_2}$ , and*

$$\|X_{P_\nu}\|_{r_\nu, q, D(s_\nu, r_\nu) \times \mathcal{O}_\nu}^{\lambda_\nu} \leq \varepsilon_\nu.$$

*Then there exists a family of symplectic coordinate transformation*

$$\Phi_{\nu+1} : D_{\nu+1} \times \mathcal{O}_\nu \longrightarrow D_\nu$$

*and a closed subset*

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \bigcup_{|k| > 0, l} \mathfrak{R}_{k, l}^{\nu+1}(\gamma_{\nu+1}),$$

where

$$\begin{aligned}
\mathfrak{N}_{k,l}^{\nu+1}(\gamma_{\nu+1}) &= \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle| < \frac{\gamma_{\nu+1}}{|k|^\tau}; \\
|\langle k, \omega_{\nu+1}(\xi) \rangle \pm \Omega_j^{\nu+1}(\xi)| &< \frac{\gamma_{\nu+1}|j|^{1+\delta}}{|k|^\tau}; \\
|\langle k, \omega_{\nu+1}(\xi) \rangle \pm \Omega_i^{\nu+1}(\xi) \pm \Omega_j^{\nu+1}(\xi)| &< \frac{\gamma_{\nu+1}(|i| + |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}; \\
|\langle k, \omega_{\nu+1}(\xi) \rangle + \Omega_i^{\nu+1}(\xi) - \Omega_j^{\nu+1}(\xi)| &< \frac{\gamma_{\nu+1}(|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, |i| \neq |j|; \\
|\langle k, \omega_{\nu+1}(\xi) \rangle + \Omega_j^{\nu+1}(\xi) - \Omega_{-j}^{\nu+1}(\xi)| &< \frac{\gamma_{\nu+1}|j|^\delta}{|k|^\tau}. \}
\end{aligned}$$

such that for  $H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1}$  the same assumptions as above are satisfied with “ $\nu + 1$ ” in place of “ $\nu$ ”.

*Proof.* By induction one verifies that  $\varepsilon_\nu \leq \frac{\gamma_\nu \sigma_\nu^{2\tau+6}}{C^3}$ . With the definition of  $\eta_\nu$ , namely  $\eta_\nu^3 = \frac{\varepsilon_\nu}{\gamma_\nu \sigma_\nu^{2\tau+3}}$ , this implies  $\varepsilon_\nu \leq \frac{\gamma_\nu \sigma_\nu^{2\tau+4} \eta_\nu^2}{C}$ . By the KAM step there exists a transformation  $\Phi_{\nu+1} : D_{\nu+1} \times \mathcal{O}_\nu \rightarrow D_\nu$  taking  $H_\nu$  into  $H_{\nu+1} = N_{\nu+1} + P_{\nu+1}$ . The new perturbation  $P_{\nu+1}$  then satisfies the estimate

$$\begin{aligned}
\|X_{P_{\nu+1}}\|_{r_{\nu+1}, q, D(s_{\nu+1}, r_{\nu+1}) \times \mathcal{O}_{\nu+1}}^{\lambda_{\nu+1}} &\leq C \eta_\nu \varepsilon_\nu + \frac{C}{\gamma_\nu \sigma_\nu^{2\tau+3} \eta_\nu^2} \varepsilon_\nu^2 \\
&= C(\gamma_\nu \sigma_\nu^{2\tau+3})^{-\frac{1}{3}} \varepsilon_\nu^{\frac{4}{3}} = \varepsilon_{\nu+1}.
\end{aligned}$$

In view of (5.12) the Lipschitz semi-norm of the new frequencies is bounded by

$$M_\nu + C \|X_{P_\nu}\|_{r_\nu, q, D(s_\nu, r_\nu) \times \mathcal{O}_\nu}^{\lambda_\nu} \leq M_\nu + \frac{C \varepsilon_\nu}{\gamma_\nu} M_\nu \leq M_\nu (1 + 2^{-\nu-2}) \leq M_{\nu+1}$$

as required. By the estimate (5.13), that is,

$$|\langle l, \Omega^{\nu+1} - \Omega^\nu(\xi) \rangle| \leq \varepsilon_\nu \langle l \rangle_{d-1} \leq \frac{\gamma_\nu \sigma_\nu^{2\tau+6}}{C^3} \langle l \rangle_{d-1} \leq \frac{m_0}{2^{\nu+2}} \langle l \rangle_{d-1},$$

then  $|\langle l, \Omega^{\nu+1}(\xi) \rangle| \geq m_{\nu+1} \langle l \rangle_{d-1}$  for  $0 < |l| \leq 2$ .  $\square$

## 5.4 Convergence

Let

$$\Psi^\nu = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_\nu : D_\nu \times \mathcal{O}_{\nu-1} \rightarrow D_0.$$

Inductively, we have that

$$H_\nu = H \circ \Psi^\nu = \langle \omega_\nu(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j^\nu(\xi) z_j \bar{z}_j + P_\nu(\theta, I, z, \bar{z}, \xi).$$

Note that  $\varepsilon_\nu \rightarrow 0$  as  $\nu \rightarrow +\infty$ , we can make the KAM estimates go on well at each step. Let  $\mathcal{O}_* = \bigcap_{\nu=0}^\infty \mathcal{O}_\nu$ . As in [27], thanks to Lemma 5.5, it concludes that  $N_\nu$ ,  $\omega_\nu$ ,  $\Omega^\nu$ ,  $\Psi^\nu$  and  $D\psi^\nu$  converge uniformly on  $D(\frac{s}{2}, 0) \times \mathcal{O}_*$  with

$$N_\infty = \langle \omega_\infty(\xi), I \rangle + \sum_{j \in \mathbb{Z}_1} \Omega_j^\infty(\xi) z_j \bar{z}_j.$$



Let  $X_H^t$  be the flow of  $X_H$ . Since  $H_\nu = H \circ \Psi^\nu$ , we have

$$X_H^t \circ \Psi^\nu = \Psi^\nu \circ X_{H_\nu}^t. \quad (5.15)$$

The uniform convergence of  $\Psi^\nu$ ,  $D\Psi_\nu$ ,  $X_{H_\nu}^t$  implies that the limits can be taken on the both sides of (5.15). Hence, on  $D(\frac{s}{2}, 0) \times \mathcal{O}_*$ , we get

$$X_H^t \circ \Psi^\infty = \Psi^\infty \circ X_{H_\infty}^t. \quad (5.16)$$

and

$$\Psi^\infty : D(\frac{s}{2}, 0) \times \mathcal{O}_* \longrightarrow D(s, r) \times \mathcal{O}, \quad (5.17)$$

it follows from (5.17) that we get an invariant finite dimensional tori  $\Psi^\infty(\mathbb{T}^2 \times \{\xi\})$  for the original perturbed Hamiltonian system at  $\xi \in \mathcal{O}$ . We remark that the frequencies  $\omega_*(\xi) = \omega_\infty(\xi)$  associated with  $\Psi^\infty(\mathbb{T}^2 \times \{\xi\})$  are slightly deformed from the unperturbed ones  $\omega(\xi)$ . The normal behaviors of the invariant tori  $\Psi^\infty(\mathbb{T}^2 \times \{\xi\})$  are governed by their respective normal frequencies  $\Omega_n^\infty$ .

## 5.5 Measure Estimate

For each  $|k| > 0$ , we denote

$$\mathcal{R}_k^{\nu+1} = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle| < \frac{\gamma_{\nu+1}}{|k|^\tau}\},$$

$$\mathcal{R}_{kj}^{\nu+1} = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle \pm \Omega_j^{\nu+1}(\xi)| < \frac{\gamma_{\nu+1}|j|^{1+\delta}}{|k|^\tau}\},$$

$$\mathcal{R}_{kij}^{\nu+1} = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle \pm \Omega_i^{\nu+1}(\xi) \pm \Omega_j^{\nu+1}(\xi)| < \frac{\gamma_{\nu+1}(|i| + |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}\},$$

$$\overline{\mathcal{R}}_{kij}^{\nu+1} = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle + \Omega_i^{\nu+1}(\xi) - \Omega_j^{\nu+1}(\xi)| < \frac{\gamma_{\nu+1}(|i| - |j|)(|i|^\delta + |j|^\delta)}{|k|^\tau}, \quad |i| \neq |j|\},$$

$$\overline{\mathcal{R}}_{kj(-j)}^{\nu+1} = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle + \Omega_j^{\nu+1}(\xi) - \Omega_{-j}^{\nu+1}(\xi)| < \frac{\gamma_{\nu+1}|j|^\delta}{|k|^\tau}\},$$

then

$$\mathfrak{R}_{k,l}^{\nu+1}(\gamma_{\nu+1}) = \mathcal{R}_k^{\nu+1} \cup \bigcup_{i,j} \left( \mathcal{R}_{kj}^{\nu+1} \cup \mathcal{R}_{kij}^{\nu+1} \cup \overline{\mathcal{R}}_{kij}^{\nu+1} \right) \cup \bigcup_{|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\}|k|} \overline{\mathcal{R}}_{kj(-j)}^{\nu+1}.$$

At each step, we have to exclude the following resonant set:

$$\mathfrak{R}^{\nu+1} = \bigcup_{|k| > 0, l} \mathfrak{R}_{k,l}^{\nu+1}(\gamma_{\nu+1}),$$

then

$$\mathcal{O} \setminus \mathcal{O}_* = \bigcup_{\nu \geq 0} \mathfrak{R}^{\nu+1}.$$

Note that

$$\mathfrak{R}_{k,l}^{\nu+1} \setminus \bigcup_{|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\}|k|} \overline{\mathcal{R}}_{kj(-j)}^{\nu+1} \subset \widetilde{\mathfrak{R}}_{k,l}^{\nu+1},$$

where

$$\widetilde{\mathfrak{R}}_{k,l}^{\nu+1}(\gamma_{\nu+1}) = \{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi) \rangle + \langle l, \Omega^{\nu+1}(\xi) \rangle| < \frac{\gamma_{\nu+1} \langle l \rangle^\delta}{|k|^\tau}\}.$$

Now we will prove that the measure of set  $\widetilde{\mathfrak{R}}_{k,l}^{\nu+1}$  is small, so does  $\mathfrak{R}_{k,l}^{\nu+1}$ .

**Lemma 5.8.** *If  $\tilde{\mathfrak{R}}_{k,l}^\nu(\gamma_\nu) \neq \emptyset$ , then*

$$\langle l \rangle_{d-1} \leq c|k|,$$

where  $c = 4(1 + |\omega|_{\mathcal{O}}^{\text{sup}})/m$  is independent of  $\nu$ .

*Proof.* If there exists  $\xi \in \tilde{\mathfrak{R}}_{k,l}^\nu(\gamma_\nu)$ , then (5.14) implies that for  $k \neq 0$ ,

$$\begin{aligned} |\langle k, \omega_\nu(\xi) \rangle| &\geq |\langle l, \Omega^\nu(\xi) \rangle| - \gamma_\nu \frac{\langle l \rangle_\delta}{|k|^\tau}, \\ &\geq m_\nu \langle l \rangle_\delta - \gamma_\nu \langle l \rangle_\delta, \\ &\geq \frac{m}{4} \langle l \rangle_{d-1} \end{aligned}$$

since  $\langle l \rangle_\delta \leq \langle l \rangle_{d-1}$  for  $\delta \leq d-1$  and  $\gamma_\nu \leq \frac{m_\nu}{2}$ ,  $m_\nu \geq \frac{m}{2}$  by construction. Hence,

$$\frac{m}{4} \langle l \rangle_{d-1} \leq |k| |\omega_\nu(\xi)| \leq |k| (1 + |\omega|_{\mathcal{O}}^{\text{sup}}).$$

□

**Lemma 5.9.** *For fixed  $\nu+1$ ,  $k$ ,  $l$ ,*

$$\text{meas } \tilde{\mathfrak{R}}_{k,l}^{\nu+1}(\gamma_{\nu+1}) < C \rho_\nu \frac{\gamma_{\nu+1}}{|k|^{\tau+1}},$$

where  $\rho_\nu$  is the diameter of  $\mathcal{O}_\nu$ .

*Proof.* Denote

$$f(\xi) = \langle k, \omega_{\nu+1}(\xi) \rangle + \langle l, \Omega^{\nu+1}(\xi) \rangle,$$

let vector  $\nu$  satisfy  $\langle k, \nu \rangle = |k|$ . It follows that

$$\frac{df(\xi + t\nu)}{dt} \geq C|k| > 0,$$

where  $C$  is some positive constant. Then by using Lemma A.6, it is easy to prove that estimate. So we omit it here. □

**Lemma 5.10.** *For fixed  $\nu+1$ ,  $k$ ,  $l$ ,*

$$\text{meas} \left( \bigcup_{|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\}} \overline{\mathcal{R}}_{kj(-j)}^{\nu+1} \right) \leq C \rho_\nu \frac{\gamma_{\nu+1}}{|k|^\tau}$$

*Proof.* Like Lemma 5.9, we have that

$$\left| \frac{\partial(\langle k, \omega_{\nu+1}(\xi) \rangle + \Omega_j^{\nu+1}(\xi) - \Omega_{-j}^{\nu+1}(\xi))}{\partial \xi} \right| \geq C|k| > 0.$$

By Lemma A.6, we know

$$\text{meas } \overline{\mathcal{R}}_{kj(-j)}^{\nu+1} \leq C \rho_\nu |j|^\delta \frac{\gamma_{\nu+1}}{|k|^{\tau+1}}.$$

Since  $|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\} |k|$  and  $\delta = 1$ , we obtain

$$\text{meas} \left( \bigcup_{|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\}} \overline{\mathcal{R}}_{kj(-j)}^{\nu+1} \right) \leq C \rho_\nu \frac{\gamma_{\nu+1}}{|k|^\tau}.$$

□

**Lemma 5.11.** For fixed  $\nu + 1 \geq 0$ ,

$$\begin{aligned} \text{meas} \left( \bigcup_{|k| > 0, l} \tilde{\mathfrak{R}}_{k,l}^{\nu+1}(\gamma_{\nu+1}) \right) &\leq C \rho_\nu \gamma_{\nu+1}, \\ \text{meas} \left( \bigcup_{|k| > 0, l} \bigcup_{|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\}} \overline{\mathcal{R}}_{kj(-j)}^{\nu+1} \right) &\leq C \rho_\nu \gamma_{\nu+1}, \end{aligned}$$

where  $C$  is a constant.

*Proof.* For a fixed  $k$ , it suffices to consider  $l$  with  $\langle l \rangle_{d-1} \leq c|k|$  according to Lemma 5.8. Taking into account that  $|l|_{d-1} \leq 2\langle l \rangle_{d-1}$ , we get

$$\text{card}\{l : |l| \leq 2, \langle l \rangle_{d-1} \leq c|k|\} \leq c|k|^s, \quad s = \frac{2}{d-1}.$$

Hence, by Lemma 5.9 and 5.10,

$$\text{meas} \left( \bigcup_l \tilde{\mathfrak{R}}_l^{\nu+1}(\gamma_{\nu+1}) \right) \leq C \rho_\nu \frac{\gamma_{\nu+1}}{|k|^{\tau-s}}.$$

If we choose  $\tau \geq s + 3$ , then

$$\text{meas} \left( \bigcup_{|k| > 0, l} \tilde{\mathfrak{R}}_{k,l}^{\nu+1}(\gamma_{\nu+1}) \right) \leq C \rho_\nu \gamma_{\nu+1}.$$

Similarly, we can prove the second measure estimate. So, Lemma 5.11 follows.  $\square$

By Lemma 5.11, we can obtain the following result about the finite dimension Lebesgue measure of  $(\mathcal{O}_\nu \setminus \mathcal{O}_{\nu+1})$ , i.e.,

$$\begin{aligned} \text{meas}(\mathcal{O}_\nu \setminus \mathcal{O}_{\nu+1}) &= \text{meas} \left( \bigcup_{|k| > 0, l} \mathfrak{R}_{k,l}^{\nu+1}(\gamma_{\nu+1}) \right) \\ &\leq \text{meas} \left( \bigcup_{|k| > 0, l} \tilde{\mathfrak{R}}_{k,l}^{\nu+1}(\gamma_{\nu+1}) \right) + \text{meas} \left( \bigcup_{|k| > 0, l} \bigcup_{|j| \leq \frac{1}{2} \max\{|n_1|, |n_2|\}} \overline{\mathcal{R}}_{kj(-j)}^{\nu+1} \right) \\ &= O(\gamma_{\nu+1}) \longrightarrow 0, \end{aligned}$$

as  $\nu \longrightarrow \infty$ . It follows that the measure of all excluded parameters can be as small as we wish.

Finally we get a Cantor-like parameter set  $\mathcal{O}_* = \bigcap_{\nu=0}^{\infty} \mathcal{O}_\nu$  of positive Lebesgue measure.

## 6 Appendix

In this section, we give some technical lemmas.

*Lemma A.1* Generalized Cauchy inequalities

$$\|F_\theta\|_{D(s-\sigma, r)} \leq \frac{c}{\sigma} \|F\|_{D(s, r)}, \quad \|F_I\|_{D(s, \frac{r}{2})} \leq \frac{c}{r^2} \|F\|_{D(s, r)},$$

*Proof.* The proof can be found in [22], [29].  $\square$

*Lemma A.2* For  $\nu > 0, 0 < \delta < 1$ , we have

$$\sum_{k \in \mathbb{Z}^n} |k|^\nu e^{-2|k|^\delta} \leq \left(\frac{\nu}{e}\right)^\nu \frac{1}{\delta^{\nu+n}} (1+e)^n.$$

*Proof.* The inequality can be found on page 22 in [3].  $\square$

*Lemma A.3* Let  $u_j, j \geq 1$ , be complex functions on  $\mathbb{T}^n$  that are real analytic on  $D(s) = \{|Im x| < s\}$ . Then

$$\left( \sum_{j \geq 1} \sup_{x \in D(s-\sigma)} |u_j(x)|^2 \right)^{\frac{1}{2}} \leq \frac{4^n}{\sigma^n} \sup_{x \in D(s)} \left( \sum_{j \geq 1} |u_j(x)|^2 \right)^{\frac{1}{2}},$$

for  $0 < \sigma \leq s \leq 1$ .

*Proof.* The proof can be found on page 262–263 in [22].  $\square$

*Lemma A.4* Let  $A = (A_{ij})_{i,j \neq 0}$  be a bounded operator on  $\ell^2$  which depends on  $x \in \mathbb{T}^n$  such that all coefficients are analytic on  $D(s) = \{|Im x| < s\}$ . Suppose  $B = (B_{ij})_{i,j \neq 0}$  is another operator on  $\ell^2$  depending on  $x$  whose coefficients satisfy

$$\sup_{x \in D(s)} |B_{ij}(x)| \leq \frac{1}{\|i\| - \|j\|} \sup_{x \in D(s)} |A_{ij}(x)|, \quad |i| \neq |j|,$$

and  $B_{jj} = 0, B_{-jj}$  for  $j \neq 0$ . Then  $B$  is a bounded operator on  $\ell^2$  for every  $x \in D(s)$ ,

$$\sup_{x \in D(s-\sigma)} \|B(x)\| \leq \frac{4^{n+1}}{\sigma^n} \sup_{x \in D(s)} \|A(x)\|,$$

for  $0 < \sigma \leq s \leq 1$ .

*Proof.* The proof can be found on page 262–263 in [22].  $\square$

Let  $V$  be an open domain in a real Banach space  $E$  with norm  $\|\cdot\|$ ,  $\Pi$  a subset of another real Banach space, and  $X : V \times \Pi \rightarrow E$  a parameter dependent vector field on  $V$ , which is  $C^1$  on  $V$  and Lipschitz on  $B$ . Let  $\phi^t$  be its flow. Suppose there is a subdomain  $U \subset V$  such that  $\phi^t : V \times \Pi \rightarrow E$  for  $-1 \leq t \leq 1$ .

*Lemma A.5* Under the preceding assumptions,

$$\begin{aligned} \|\phi^t - id\|_U &\leq \|X\|_V, \\ \|\phi^t - id\|_U^{\text{lip}} &\leq \exp(\|DX\|_V) \|X\|_V^{\text{lip}}, \end{aligned}$$

for  $-1 \leq t \leq 1$ , where all norms are understood to be taken also over  $\Pi$ .

*Proof.* The proof can be found in [27].  $\square$

*Lemma A.6* Suppose that  $g(u)$  is a  $C^N$  function on the closure  $\bar{I}$ , where  $I \subset \mathbb{R}^1$  is an interval. Let  $I_h = \{u : |g(u)| \leq h\}, h > 0$ . If for some constant  $d > 0$ ,  $|g^N(u)| \geq d$  for  $\forall u \in I$ , then  $|I_h| \leq ch^{\frac{1}{N}}$ , where  $|I_h|$  denotes the Lebesgue measure of  $I_h$  and the constant  $c = 2(2 + 3 + \dots + N + d^{-1})$ .

*Remark:* In fact, if  $N = 1$ , then  $c = 2d^{-1}$ ; if  $N = 2$ , then  $c = 2(2 + d^{-1})$ ; if  $N \geq 3$ , then  $c = 2(2 + 3 + \dots + N + d^{-1})$ .

*Proof.* The proof can be found in [36].  $\square$

## References

- [1] D. Bambusi and S. Graffi, Time quasi-periodic unbounded perturbations of Schrödinger operators and KAM methods. *Comm. Math. Phys.* 219 (2001), 465–480.
- [2] M. Berti and L. Biasco, Branching of Cantor manifolds of elliptic tori and applications to PDEs, *Comm. Math. Phys.* 305 (2011), 741–796.
- [3] N. N. Bogoljubov, Yu. A. Mitropolskii, A. M. Samoilenko, *Methods of Accelerated Convergence in Nonlinear Mechanics*, Springer-Verlag, New York, 1976.
- [4] J. Bourgain, Quasi-periodic solutions of Hamiltonian perturbations for 2D linear Schrödinger equation, *Ann. of Math.* 148 (1998), 363–439.
- [5] J. Bourgain, *Green’s Function Estimates for Lattice Schrödinger Operators and Applications*, *Annals of Mathematics Studies*, 158, Princeton University Press, Princeton, 2005.
- [6] H. H. Chen, Y. C. Lee and C. S. Liu, Integrability of nonlinear Hamiltonian systems by inverse scattering method. *Phys. Scr.* 20 (1979), 190–492.
- [7] L. Chierchia and J. You, KAM tori for 1D nonlinear wave equations with periodic boundary conditions, *Comm. Math. Phys.* 211 (2000), 497–525.
- [8] W. Craig and C. Wayne, Newton’s method and periodic solutions of nonlinear wave equation, *Comm. Pure Appl. Math.* 46 (1993), 1409–1501.
- [9] H. L. Eliasson and S. B. Kuksin, KAM for the non-linear Schrödinger equation, *Ann. of Math.* 172 (2010), 371–435.
- [10] J. Geng and J. You, A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces, *Comm. Math. Phys.* 262 (2006), 343–372.
- [11] J. Geng and Y. Yi, Quasi-periodic solutions in a nonlinear Schrödinger equation, *J. Differential Equations* 233 (2007), 512–542.
- [12] J. Geng and J. Wu, Real analytic quasi-periodic solutions for the derivative nonlinear Schrödinger equations, *J. Math. Phys.* 53 (2012), 102702.
- [13] B. Grébert and L. Thomann, KAM for the quantum harmonic oscillator, *Comm. Math. Phys.* 307 (2011), 383–427.
- [14] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.* 19 (1978), 798–801.
- [15] Y. Kodama, Optical solitons in a monomode fiber, *J. Stat. Phys.* 39 (1985), 597–614.
- [16] S. B. Kuksin, Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum, *Funktsional. Anal. iPrilozhen.* 21 (1987), 22–37; English translation in: *Funct. Anal. Appl.* 21 (1987), 192–205.
- [17] S. B. Kuksin, Perturbations of quasiperiodic solutions of infinite-dimensional Hamiltonian systems, *Izv. Akad. Nauk SSSR Ser. Mat.* 52 (1988), 41–63; English translation in: *Math. USSR-Izv.* 32 (1989), 39–62.
- [18] S. B. Kuksin, *Nearly Integrable Infinite-Dimensional Hamiltonian Systems*, *Lecture Notes in Mathematics*, 1556, Springer-Verlag, Berlin, 1993.

- [19] S. B. Kuksin and J. Pöschel, Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. of Math.* 143 (1996), 147–79.
- [20] S. B. Kuksin, On small-denominators equations with large variable coefficients, *Z. Angew. Math. Phys.* 48 (1997), 262–271.
- [21] S. B. Kuksin, *Analysis of Hamiltonian PDEs*. Oxford University Press, Oxford, 2000.
- [22] T. Kappeler and J. Pöschel, *KdV and KAM*. Springer-Verlag, Berlin, 2003.
- [23] J. Liu and X. Yuan, Spectrum for quantum Duffing oscillator and small-divisor equation with large variable coefficient. *Comm. Pure. Appl. Math.* 63 (2010), 1145–1172.
- [24] J. Liu and X. Yuan, A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations. *Comm. Math. Phys.* 307 (2011), 629–673.
- [25] J. Liu and X. Yuan, KAM for the derivative nonlinear Schrödinger equation with periodic boundary conditions. *J. Differential Equations* 256 (2014), 1627–1652.
- [26] E. Mjølhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, *J. Plasma Phys.* 16 (1976), 321–334.
- [27] J. Pöschel, A KAM theorem for some nonlinear partial differential equations, *Ann. Scuola Norm. Sup. Pisa. Cl. Sci.* 23 (1996), 119–148.
- [28] J. Pöschel, Quasi-periodic solutions for nonlinear wave equations, *Comment. Math. Helv.* 71 (1996), 269–296.
- [29] J. Pöschel, A lecture on the classical KAM theorem. *Proc. Sympos. Pure Math.* 69 (2001) 707–732.
- [30] J. Si, Quasi-periodic solutions of a non-autonomous wave equations with quasi-periodic forcing, *J. Differential Equations* 252 (2012), 5274–5360.
- [31] M. Wadati and K. Sogo, Gauge transformations in soliton theory. *J. Phys. Soc. Jpn.* 52 (1983), 394–338.
- [32] C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Comm. Math. Phys.* 127 (1990), 479–528.
- [33] S. Xu, J. He and L. Wang, Two kinds of rogue waves of the general nonlinear Schrödinger equation with derivative, preprint, arXiv:1202.0356.
- [34] X. Yuan, Quasi-periodic solutions of nonlinear Schrödinger equations of higher dimension, *J. Differential Equations* 195 (2003), 230–242.
- [35] X. Yuan, Quasi-periodic solutions of completely resonant non linear wave equations, *J. Differential Equations* 230 (2006), 213–274.
- [36] J. You, A KAM theorem for hyperbolic-type degenerate lower dimensional tori in Hamiltonian systems, *Comm. Math. Phys.* 192 (1998), 145–168.
- [37] M. Zhang and J. Si, Quasi-periodic solutions of nonlinear wave equations with quasi-periodic forcing, *Phys. D* 238 (2009), 2185–2215.